LAURENT SERIES

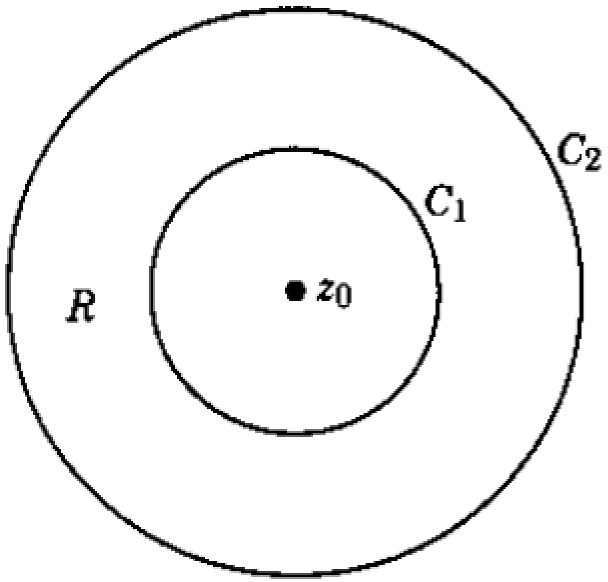
Acknowledgement

- Mathematical Methods in the Physical Sciences – Mary L. Boas

Laurent's theorem: Let C_1 and C_2 be the two circles with centre at z_0. Let $f(z)$ be analytic in region R between the circles. Then $f(z)$ can be expanded in a series of the form

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ... + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + ...$$

convergent in R. Such a series is called a Laurent series. The 'b' series is called the principal part of the Laurent series.



Consider the Laurent series

$$f(z) = 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots + \left(\frac{z}{2}\right)^n + \ldots + \frac{2}{z} + 4\left(\frac{1}{z^2} - \frac{1}{z^3} + \ldots + \frac{(-1)^n}{z^n} + \ldots\right)$$

From ratio test of the positive powers of the series, series converges for $|\frac{z}{2}| < 1$, that is for $|z| < 2$. Similarly, convergence of negative powers is for $|\frac{1}{z}| < 1$, that is $|z| > 1$. The Laurent series converges (both negative and positive power of the series) is for $|z|$ between 1 and 2, that is in an annular region between two circles C_1 and C_2 of radii 1 and 2. The 'a' series converges inside circle C_2 and 'b' series (inverse power of z) converges outside circle C_1. The Laurent series converges between two circles, C_1 and C_2.

The formulas for the coefficients

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}$$

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}}$$

where C is any simple closed curve surrounding z_0 and lying in R.

Definitions:

If all the b's are zero, $f(z)$ is analytic at $z = z_0$, and we call z_0 a regular point.

If $b_n \neq 0$, but all b's after b_n are zero, $f(z)$ is said to have a pole of order n at $z = z_0$. If $n = 1$, we say that $f(z)$ has a simple pole.

If there are an infinite number of b's different from zero, $f(z)$ has an essential singularity at $z = z_0$.

The coefficient b_1 of $1/(z - z_0)$ is called the residue of $f(z)$ at $z = z_0$.

For the following function find the first few terms of each of the Laurent series about the origin, one series for each annular ring between singular points. Also find the residue of the function at the origin.

$$\frac{1}{z(z-1)(z-2)}$$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots$$

$$\frac{1}{z-1} = -(1-z)^{-1} = -(1 + z + z^2 + z^3 + \ldots)$$

$$\frac{1}{z-2} = -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots\right)$$

Partial Fractions

It is often convenient to write a fraction as a sum of two simpler fractions; for example

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

To find A and B, we first clear the fractions and deal with the identity
$1 = A(z-2) + B(z-1)$
At $z = 2$, $B = 1$ and at $z = 1$, $A = -1$. Thus

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$f(z) = \frac{1}{z}\frac{1}{(z-1)(z-2)} = \frac{1}{z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right)$$

There are three regions each with a different Laurent series.
(Region I): $0 < |z| < 1$. In this region we use only powers of z.

$$\frac{1}{z}\frac{1}{(z-1)(z-2)} = \frac{1}{z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right)$$

$$= -\frac{1}{2z}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots\right) + \frac{1}{z}\left(1 + z + z^2 + z^3 + \ldots\right)$$

$$= \frac{1}{2z} - \frac{1}{2} + \frac{7z}{8} + \frac{15z^2}{16} + \ldots$$

(Region I): $0 < |z| < 1$

$$\frac{1}{z}\frac{1}{(z-1)(z-2)} = \frac{1}{z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right)$$

$$= \frac{1}{2z} - \frac{1}{2} + \frac{7z}{8} + \frac{15z^2}{16} + \ldots$$

The series converges near $z = 0$; the residue of the given function at the origin is the coefficient of $1/z$ in this series, that is $R = \frac{1}{2}$.

(Region II):In the region $1 < |z| < 2$, we use the series in powers of z for $1/(z-2)$ since it converge for $|z| < 2$, and the series in powers of $1/z$ for $1/(z-1)$ since this converges for $|z| > 1$.

$$\frac{1}{z}\frac{1}{(z-1)(z-2)} = \frac{1}{z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right)$$

$$= \frac{1}{z}\left[\frac{1}{z-2} - \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right)\right]$$

$$= \frac{1}{z}\left(\frac{1}{z-2}\right) - \frac{1}{z^2}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2z}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \frac{z^4}{16} + ...\right) - \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ...\right)$$

$$= -\left(...z^{-4} + z^{-3} + z^{-2} + \frac{z^{-1}}{2} + \frac{1}{4} + \frac{z}{8} + \frac{z^2}{16} + \frac{z^3}{32} + ...\right)$$

This series is an expansion about $z = 0$ (that is, in powers of z). Note carefully, however, that the point $z = 0$ is not in the region of convergence. This series cannot be used to find the residue at the origin. The coefficient of $1/z$ in this series is <u>not</u> the residue at the origin.

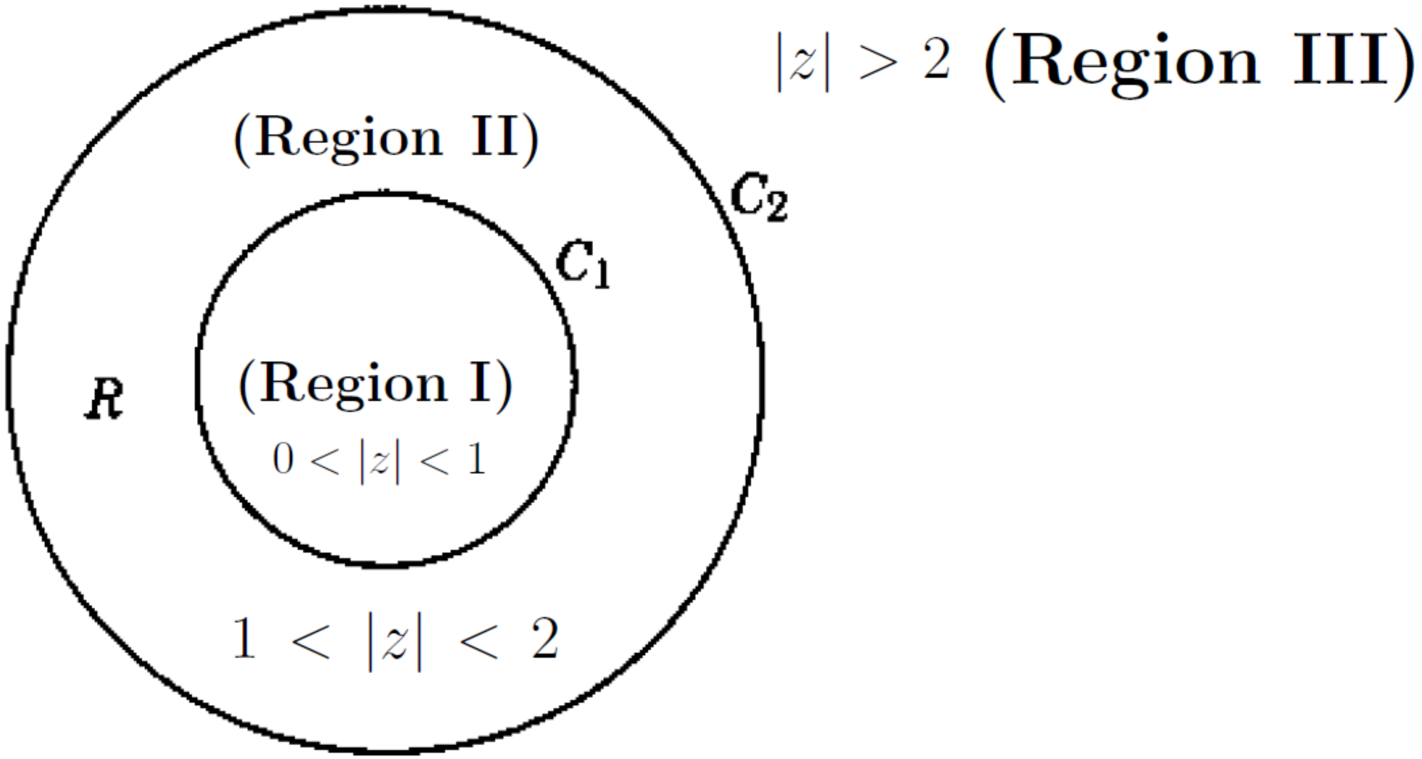
(Region III): In the region $|z| > 2$, we use only the series of powers of $1/z$ since all these converge for $|z| > 2$.

$$\frac{1}{z}\frac{1}{(z-1)(z-2)} = \frac{1}{z}\left(\frac{1}{z-2} - \frac{1}{z-1}\right)$$

$$= \frac{1}{z}\left[\frac{1}{z}\left(\frac{1}{1-\frac{2}{z}}\right) - \frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right)\right]$$

$$= \frac{1}{z^2}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \ldots\right) - \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\right)$$

$$= \left(\frac{1}{z^3} + \frac{3}{z^4} + \frac{7}{z^5} + \ldots\right)$$


$|z| > 2$ (Region III)
(Region II)
C_2
C_1
R
(Region I)
$0 < |z| < 1$
$1 < |z| < 2$

For the following function find the first few terms of each of the Laurent series about the origin, one series for each annular ring between singular points. Also find the residue of the function at the origin.

$$\frac{1}{z(z-1)(z-2)^2}$$

For the following function find the first few terms of each of the Laurent series about the origin, one series for each annular ring between singular points. Also find the residue of the function at the origin.

$$\frac{2-z}{1-z^2}$$

For the following function find the first few terms of each of the Laurent series about the origin, one series for each annular ring between singular points. Also find the residue of the function at the origin.

$$\frac{30}{(1+z)(z-2)(3+z)}$$