The Residue Theorem

Acknowledgement

- Mathematical Methods in the Physical Sciences – Mary L. Boas

Our task is to find the value of $\oint_c f(z)dz$ around a simple closed curve C surrounding z_0 as an isolated singular point of $f(z)$. If we expand f(z) in Laurent series about $z = z_0$ that converges near $z = 0$, we find that taking integral of the 'a' series terms reduces to zero (as they are analytic functions of z) following Cauchy's theorem. While evaluating the 'b' series terms, we can prove that all other b_n terms are zero except b_1. So the integral of the function $f(z)$ i.e. $\oint_c f(z)dz$ is equal to $2\pi i b_1$. b_1 is called the residue of $f(z)$ at $z = z_0$.

$$\oint_c f(z)dz = \oint_c a_0 dz + \oint_c a_1(z-z_0)dz + \oint_c a_2(z-z_0)^2 dz + \ldots$$

$$+\oint_c \frac{b_1}{z-z_0}dz + \oint_c \frac{b_2}{(z-z_0)^2}dz + \ldots$$

$$\sum_n \oint_c a_n(z-z_0)^n dz = 0$$

as $(z-z_0)^n$ $(n \geq 0)$ are analytic functions, so all integral of 'a' series is zero.

Taking $z = z_0 + \rho e^{i\theta}$

$$\oint_c \frac{b_1}{z-z_0}dz = b_1 \int_0^{2\pi} \frac{\rho i e^{i\theta} d\theta}{\rho e^{i\theta}} = 2\pi i b$$

$$\oint_c \frac{b_n}{(z-z_0)^n} dz = 0 \qquad \text{if } n \neq 1$$

$$\oint_c f(z) dz = 2\pi i \cdot \text{residue of } f(z) \text{ at the singular point inside } C$$

If there are several singularities inside C, small circles are drawn about each singularities $(z_0, z_1, z_2, \cdots)$ so that $f(z)$ is analytic in the region between C and the circles.

$$\oint_c f(z)dz = 2\pi i \cdot \text{sum of the residues of } f(z) \text{ of all the singular points inside } C$$

where the integral around C is in the counterclockwise direction

Summary

The Residue Theorem: When $f(z)$ has an isolated singularity at z_0, if $f(z)$ is **analytic** in the *punctured* disk $\{0 < |z - z_0| < r\}$ centered at z_0. In that case $f(z)$ has a Laurent series representation

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots$$

in this punctured disk. The representation is unique.

The residue of $f(z)$ at z_0 is $Res(f, z_0) = b_1$, the coefficient of the term $\dfrac{1}{z - z_0}$

Question-1

If C is a circle of radius ρ about z_0, show that

$$\oint_C \frac{dz}{(z-z_0)^n} = 2\pi i \qquad \text{if} \quad n = 1,$$

but for any other integral value of n, positive or negative, the integral is zero. Use the fact that $z = z_0 + \rho e^{i\theta}$ on C.

Question-2

Verify the formulas

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}, \qquad b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}},$$

for the coefficients in a Laurent series.

Question-3

Obtain Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$

from the residue theorem

$$\oint_c f(z)dz = 2\pi i \cdot \text{sum of the residues of } f(z) \text{ of all the singular points inside } C$$