Contour Integrals

Acknowledgement

- Mathematical Methods in the Physical Sciences – Mary L. Boas

Cauchy's Theorem: Let C be a simple (simple curve is one which does not cross itself) closed curve with a continuously turning tangent except possibly at a finite number of points (we allow a finite number of corners otherwise we have a smooth curve). If $f(z)$ is analytic on and inside C, then

$$\oint_{around\ C} f(z)dz = 0$$

Green's theorem in the plane

$$\oint_C P\,dx + Q\,dy = \iint\limits_{\substack{\text{area}\\ \text{inside } C}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\oint_C (u\,dx - v\,dy) = \iint\limits_{\substack{\text{area}\\ \text{inside } C}} \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+i\,dy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)$$

These two line integrals are zero.

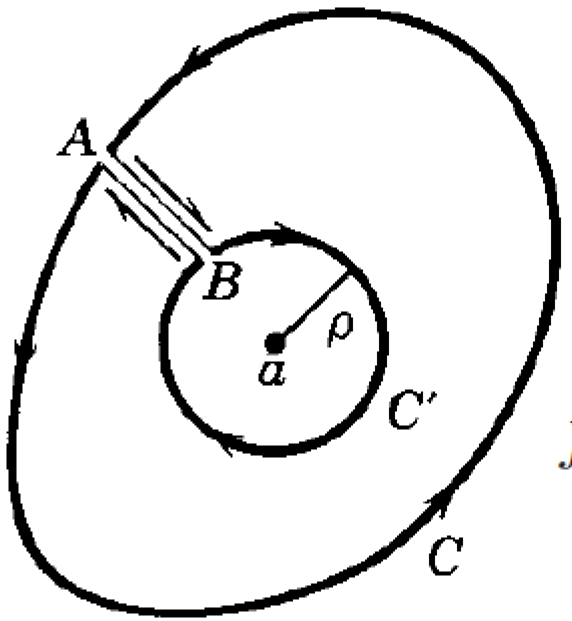
Applying Green's theorem in the plane.

Using Cauchy-Riemann conditions on the derivatives of the integrand.

Cauchy's integral formula: If $f(z)$ is analytic on and inside a simple closed C, the value of $f(z)$ at a point $z = a$ inside C is is given by the following contour integral along C:

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - a} dz$$

$$\phi(z) = \frac{f(z)}{z-a}$$



$f(z)$ is analytic on and inside C

$$\oint_{C \text{ counter-clockwise}} \phi(z)\,dz + \oint_{C' \text{ clockwise}} \phi(z)\,dz = 0$$

$$\oint_C \phi(z)\,dz = \oint_{C'} \phi(z)\,dz$$ ➡ both are counterclockwise.

Along the circle C', $z = a + \rho e^{i\theta}$

$$\oint_C \phi(z)\,dz = \oint_{C'} \phi(z)\,dz \qquad dz = \rho i e^{i\theta}\,d\theta$$

$$= \oint_{C'} \frac{f(z)}{z-a}\,dz$$

$$= \int_0^{2\pi} \frac{f(z)}{\rho e^{i\theta}} \rho i e^{i\theta}\,d\theta$$

$$= \int_0^{2\pi} f(z) i\,d\theta. \qquad \rho \to 0 \text{ (that is, } z \to a)$$

$$= \int_0^{2\pi} f(a) i\,d\theta \qquad \lim_{z\to a} f(z) = f(a)$$

$$= 2\pi i f(a)$$

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$

Hints for solving line integrals in complex plane

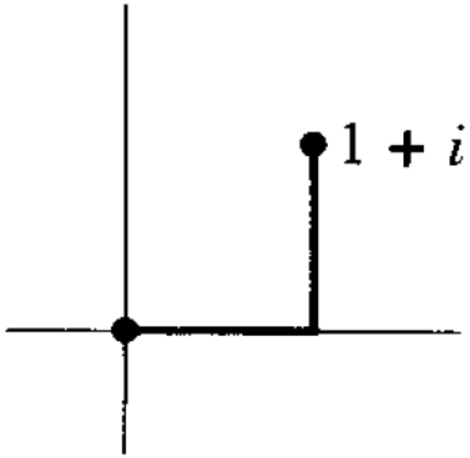
For solving line integrals in complex plane:

1. Apply line integration methods learned in multi-variable calculus.
2. Check your result by using the Cauchy's theorem.

$\int_0^{1+i} (z^2 - z)\, dz$

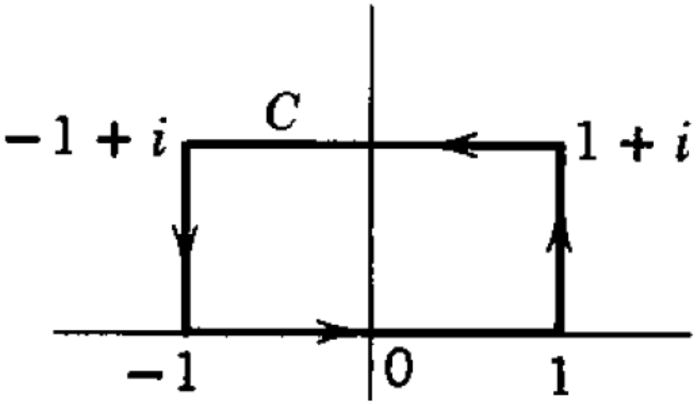
(a) along the line $y = x$;

(b) along the indicated broken line.

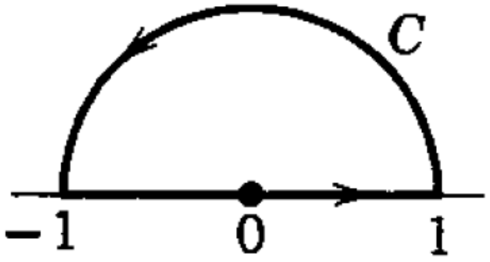


$\oint_C z^2\, dz$ along the indicated paths:

(a)

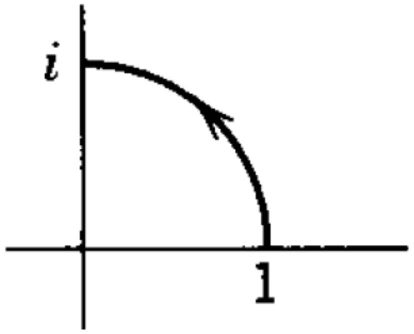


(b)



$\int_1^i z\,dz$ along the indicated paths:

(a)



(b)