5.7 Velocity of Propagation of Plane Longitudinal Waves in an Elastic Fluid

We shall derive here an expression for the velocity of plane longitudinal waves propagating in an elastic fluid medium on the basis of the following assumptions :

(i) The medium is homogeneous and isotropic.

(ii) Dissipative forces originating from viscosity and thermal conduction are absent.

(iii) The effect of gravity is negligible, so that, in equilibrium, the pressure and the density are the same everywhere in the medium.

(iv) The strain produced by the wave in the medium is so small that Hooke's law holds.

We consider a cylinder of the fluid of cross sectional area α, the axis of the cylinder coinciding with the direction of propagation of the wave. Let A_1 and B_1 be two closely spaced transverse plane sections of the cylinder (Fig. 5.3). Suppose that x and $x + \delta x$ be the equilibrium positions of the planes A_1 and B_1 with respect to an arbitrarily chosen origin, δx being much smaller than the wavelength λ of the propagating wave. The particles on the planes A_1 and B_1 are displaced due to the excess pressure produced by the progressive longitudinal wave. On the elapse of a short time interval δt (which is much smaller than the period T of the wave), let the particles on the plane A_1 be displaced parallel to the cylinder axis by ξ to A_2, where $\xi \ll \delta x$. The corresponding displacement of the particles on B_1 in time δt is B_1B_2, where $B_1B_2 = \xi + \delta\xi = \xi + \frac{\partial \xi}{\partial x}\delta x$.

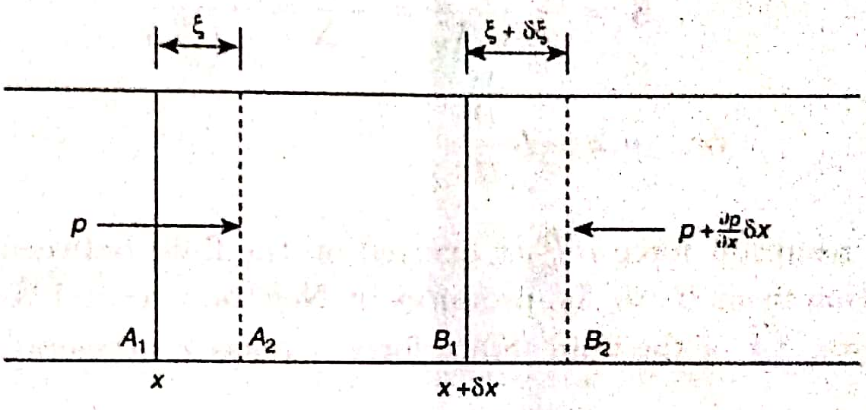

Fig. 5.3 A fluid cylinder with transverse plane sections

The displacements of the particles change the volume of the fluid between the planes. The initial volume V_0 of the fluid between A_1 and B_1 is

$$V_0 = \alpha\delta x. \tag{5.17}$$

The final volume of the fluid between A_2 and B_2 is

$$V_f = \alpha(\delta x + \delta\xi) = \alpha\left(\delta x + \frac{\partial \xi}{\partial x}\delta x\right). \tag{5.18}$$

So, the increase in volume is $\delta V = V_f - V_0 = \alpha\frac{\partial \xi}{\partial x}\delta x$. The ratio of the increase in volume δV to the initial volume V_0 is defined to be the *dilatation* or the volume strain Δ. Thus

$$\Delta = \frac{\delta V}{V_0} = \frac{\partial \xi}{\partial x}. \tag{5.19}$$

When a layer of a fluid is compressed by a disturbance, the pressure in the layer increases from the equilibrium value P_0 by an amount p, referred to as the *excess pressure*. When the layer suffers a rarefaction, p is negative. For sound waves, p is known as the *sound pressure* or the *acoustic pressure*.

The mass of the fluid between the planes A_1 and B_1 is the same as that between the planes A_2 and B_2. This mass is $\rho_0\alpha\delta x$, where ρ_0 is the equilibrium fluid density. The acoustic pressure difference between A_1 and A_2 is negligible since $\xi \ll \delta x$. Therefore, we can take the excess pressure at either A_1 or A_2 to be p, and that at either B_1 or B_2 as $p + \frac{\partial p}{\partial x}\delta x$. These two pressures act on the fluid slab A_2B_2 in opposite directions (Fig. 5.3), and produce two effects :

(a) Equal and opposite excess pressures p develop a stress on the fluid between A_2 and B_2. The stress p, if compressional, produces the volume strain $-\frac{\delta V}{V_0}$. By Hooke's law the *bulk modulus* K is defined to be the ratio between the volume stress to the volume strain :

$$K = -\frac{p}{\delta V/V_0} = -\frac{p}{\Delta} = -\frac{p}{\left(\frac{\partial \xi}{\partial x}\right)}$$

$$\text{or,}\quad p = -K\frac{\partial \xi}{\partial x}. \tag{5.20}$$

(b) The resultant force $\alpha\frac{\partial p}{\partial x}\delta x$ exerted on the fluid between B_2 and A_2 in the direction from B_2 to A_2, produces by Newton's second law of motion, an acceleration $\frac{\partial^2 \xi}{\partial t^2}$ of the fluid. Since force = mass × acceleration, we have

$$-\alpha\frac{\partial p}{\partial x}\delta x = (\rho_0\alpha\delta x)\frac{\partial^2 \xi}{\partial t^2}, \tag{5.21}$$

the negative sign accounting for the fact that the unbalanced force is in the negative x-direction. Equation (5.21) simplifies to

$$-\frac{\partial p}{\partial x} = \rho_0\frac{\partial^2 \xi}{\partial t^2}. \tag{5.22}$$

Substituting for p from Eq. (5.20) into Eq. (5.22), we obtain

$$K\frac{\partial^2 \xi}{\partial x^2} = \rho_0\frac{\partial^2 \xi}{\partial t^2}$$

$$\text{or,}\quad \frac{\partial^2 \xi}{\partial t^2} = \frac{K}{\rho_0}\frac{\partial^2 \xi}{\partial x^2}. \tag{5.23}$$

Comparing Eq. (5.23) with Eq. (5.8), we find that ξ satisfies the differential wave equation for plane waves, the wave velocity for ξ being

$$c = \sqrt{\frac{K}{\rho_0}}. \tag{5.24}$$

Observation : The ratio of the increase in density $\delta\rho$ of a layer to the initial density ρ_0 is defined to be the condensation, s. Thus $s = \frac{\delta\rho}{\rho_0}$ or, $\delta\rho = \rho_0 s$.

$\therefore$ The final density $\rho = \rho_0 + \delta\rho$

$= \rho_0 (1+s)$.

If V_0 and V be the initial and final volumes of the fluid slab, respectively, we have $\rho V = \rho_0 V_0$, since the mass of the slab is constant.

As $V = V_0 (1+\Delta)$

we obtain, $(1+s)(1+\Delta) = 1$

or, $1 + s + \Delta = 1$ [$\because$ s and Δ both are small fraction so that $s\Delta$ is negligible].

or, $s = -\Delta$

Thus, $K = -\frac{p}{\Delta} \Rightarrow K = \frac{p}{s} \Rightarrow p = Ks = c^2 \rho_0 s$.