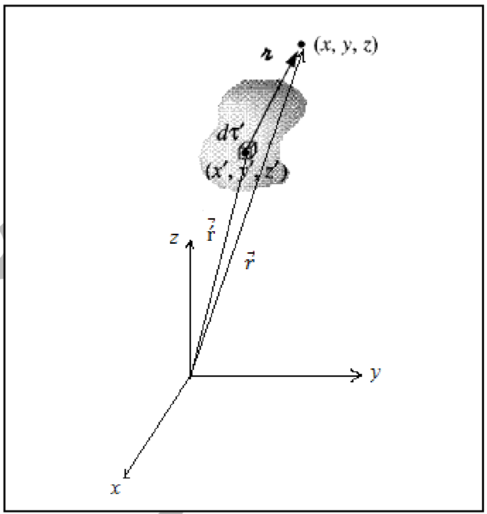
PHY-H-CC-T-03: ELECTRICY AND MAGNETISM

LECTURE-4 (Pabitra Halder (PH), Department of Physics, Berhampore Girls' College)

The Divergence and Curl of $\overrightarrow{B}$:



From Biot-Savart law, the magnetic field produced by a volume current

$\vec{B}\,(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\,(\vec{r}')\times \hat{\mathscr{r}}}{\mathscr{r}^{\,2}}\, d\tau'$, Where $\overrightarrow{\mathscr{r}} = (x\text{-}x')\,\hat{x} + (y\text{-}y')\,\hat{y} + (z\text{-}z')\,\hat{z}$.

The divergence of above magnetic field with respect to unprimed co-ordinates we get

$$\vec{\nabla}\,.\,\vec{B}\,(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{\nabla}\,.\,\left(\frac{\vec{J}\,(\vec{r}')\times \hat{\mathscr{r}}}{\mathscr{r}^{\,2}}\right) d\tau'$$

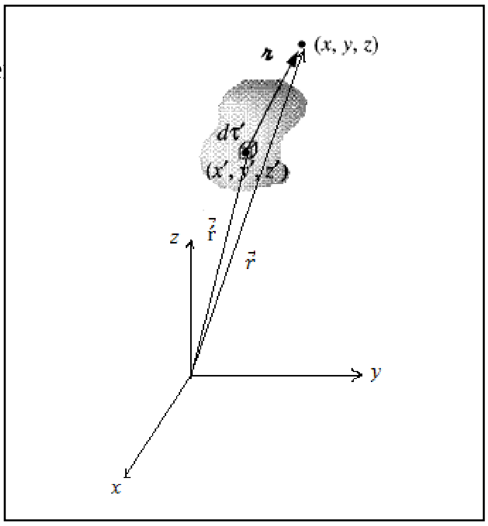
Using vector identity, $\vec{\nabla}\,.\,(\vec{A}\times\vec{B}) = \vec{B}\,.\,(\vec{\nabla}\times\vec{A}\,) - \vec{A}\,.\,(\vec{\nabla}\times\vec{B}\,)$ we get

$$\vec{\nabla}\,.\,\vec{B}\,(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^{\,2}}\,.\,(\vec{\nabla}\times\vec{J}\,(\vec{r}'))\, d\tau' - \frac{\mu_0}{4\pi}\int \vec{J}\,(\vec{r}')\,.\,\left(\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^{\,2}}\right) d\tau'$$

$\vec{\nabla}\times\vec{J}\,(\vec{r}') = 0$, since $\vec{J}\,(\vec{r}')$ does not depend on $\vec{r}$ and $\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^{\,2}} = 0$.

$$\boxed{\vec{\nabla}\,.\,\vec{B}\,(\vec{r}) = 0}$$

The Curl of $\overrightarrow{B}$:



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The divergence of above magnetic field with respect to unprimed co-ordinates we get,

$$\vec{\nabla}\times\vec{B}\,(\vec{r}) = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left(\frac{\vec{J}\,(\vec{r}')\times \hat{\mathscr{r}}}{\mathscr{r}^{\,2}}\right) d\tau'$$

Using vector identity,

$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\,.\,\vec{\nabla})\,\vec{A} - (\vec{A}\,.\,\vec{\nabla}\,)\,\vec{B} + \vec{A}\,(\vec{\nabla}\,.\,\vec{B}\,) - \vec{B}\,(\vec{\nabla}\,.\,\vec{A}\,)$ we get,

$$\vec{\nabla}\times\left(\frac{\vec{J}\,(\vec{r}')\times \hat{\mathscr{r}}}{\mathscr{r}^{\,2}}\right) = \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^{\,2}}.\,\vec{\nabla}\right)\vec{J}\,(\vec{r}') - (\vec{J}\,(\vec{r}').\,\vec{\nabla}\,)\,\frac{\hat{\mathscr{r}}}{\mathscr{r}^{\,2}} + \vec{J}\,(\vec{r}')\left(\vec{\nabla}\,.\,\frac{\hat{\mathscr{r}}}{\mathscr{r}^{\,2}}\right) - \frac{\hat{\mathscr{r}}}{\mathscr{r}^{\,2}}(\vec{\nabla}\,.\,\vec{J}\,(\vec{r}'))$$

$\vec{\nabla}\times\vec{J}\,(\vec{r}') = 0$ and $\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^{\,2}}.\,\vec{\nabla}\right)\vec{J}\,(\vec{r}') = 0$, since $\vec{J}\,(\vec{r}')$ does not depend on $\vec{r}$.

$\vec{\nabla} \times \vec{B}\,(\vec{r}) = \frac{\mu_0}{4\pi} \int \vec{J}\,(\vec{r}')\,(\vec{\nabla} \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2})\; d\tau' \;-\; \frac{\mu_0}{4\pi} \int (\vec{J}\,(\vec{r}') \cdot \vec{\nabla})\, \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\; d\tau'$. Since, $(\vec{\nabla} \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}) = 4\pi\delta^3(\vec{\mathscr{r}})$. Then

$$\vec{\nabla} \times \vec{B}\,(\vec{r}) = \mu_0 \vec{J}\,(\vec{r}) - \frac{\mu_0}{4\pi} \int (\vec{J}\,(\vec{r}') \cdot \vec{\nabla})\, \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\; d\tau' \;\cdots\cdots\cdots\; (1)$$

Now $-(\vec{J}\,(\vec{r}') \cdot \vec{\nabla})\, \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = (\vec{J}\,(\vec{r}') \cdot \vec{\nabla}')\, \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = (\vec{J}\,(\vec{r}') \cdot \vec{\nabla}')\left[\frac{(x-x')}{\mathscr{r}^3}\,\hat{x} + \frac{(y-y')}{\mathscr{r}^3}\,\hat{y} + \frac{(z-z')}{\mathscr{r}^3}\,\hat{z}\right]$

Again $(\vec{J}\,(\vec{r}') \cdot \vec{\nabla}')\, \frac{(x-x')}{\mathscr{r}^3} = \vec{\nabla}' \cdot \left[\frac{(x-x')}{\mathscr{r}^3}\, \vec{J}\,(\vec{r}')\right] - \frac{(x-x')}{\mathscr{r}^3}\,(\vec{\nabla}' \cdot \vec{J}\,(\vec{r}')) = \vec{\nabla}' \cdot \left[\frac{(x-x')}{\mathscr{r}^3}\, \vec{J}\,(\vec{r}')\right]$ (Since, for magnetostatics $\vec{\nabla}' \cdot \vec{J}\,(\vec{r}') = 0$ and Using vector identity $\vec{\nabla} \cdot (f\vec{A}) = f\,(\vec{\nabla} \cdot \vec{A}) + (\vec{A} \cdot \vec{\nabla})$)

Then $\iiint \vec{\nabla}' \cdot \left[\frac{(x-x')}{\mathscr{r}^3}\, \vec{J}\,(\vec{r}')\right] d\tau' = \oint\!\!\oint \left[\frac{(x-x')}{\mathscr{r}^3}\, \vec{J}\,(\vec{r}')\right] d\vec{a} = 0$ (For large enough integration volume, all currents are inside. So $\vec{J}\,(\vec{r}') = 0$ at the surface)

So we can write $\frac{\mu_0}{4\pi} \int (\vec{J}\,(\vec{r}') \cdot \vec{\nabla})\, \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\; d\tau' = 0$

(1)⟶ $\boxed{\vec{\nabla} \times \vec{B}\,(\vec{r}) = \mu_0 \vec{J}\,(\vec{r})}$

Magnetic Potential:

Magnetic potential is a method of representing magnetic field by using a quantity called potential instead of actual $\vec{B}$ vector field.

Magnetic Potential

Magnetic Scalar Potential | *Magnetic Vector Potential*

(1) Magnetic Scalar Potential:

In electrostatics, electric field $\vec{E}$ is derivable from the electrostatics potential V.

$\vec{\nabla} \times \vec{E} = 0 \longrightarrow \vec{E} = -\vec{\nabla}V.$

V is scalar quantity and easier to handle $\vec{E}$ which is a vector quantity.

In Magnetostatics, the quantity magnetic scalar potential can be obtain using analogous relation

$\vec{\nabla} \times \vec{B}\,(\vec{r}) = \mu_0 \vec{J}\,(\vec{r})$. In region of space in the absence of currents, the current density $\vec{J}\,(\vec{r}) = 0$. Therefore we can write $\vec{\nabla} \times \vec{B}\,(\vec{r}) = 0$. Then, $\vec{B}$ is derivable from the gradient of a potential.

Therefore $\vec{B}$ can be expressed as the gradient of a scalar quantity Φ_m: $\vec{B} = -\vec{\nabla}\,\Phi_m$.

Φ_m is called as the magnetic scalar potential.

Question:

Show that magnetic scalar potential Φ_m satisfied the Laplace's equation?

Ans. In presence of magnetic moment $\vec{m}$ creates a magnetic field $\vec{B}$, which is the gradient of some scalar field Φ_m.

The divergence of magnetic field $\vec{B}$ is zero, therefore we can write $\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0$.

By definition, the divergence of gradient of scalar field is also zero.

$-\vec{\nabla} \cdot (\vec{\nabla} \Phi_m) = 0 \rightarrow \nabla^2 \Phi_m = 0 \rightarrow$ satisfied Laplace's equation.

Laplace's equation is valid only outside the magnetic source and away from currents. Magnetic field can be calculated from the magnetic scalar potential using solution of Laplace's equation.

Note:

The magnetic scalar potential is useful only in the region of space away from the free currents. If $\vec{J}(\vec{r})=0$, then only magnetic flux density can be computed from the magnetic scalar potential.

The potential function which overcome this limitation and is useful to compute $\vec{B}$ in the region where $\vec{J}$ is present is magnetic vector potential.

Characteristics of magnetic scalar potential (Φ_m):

1. The negative gradient of Φ_m gives $\vec{B}$ or $\vec{B} = -\vec{\nabla} \Phi_m$.

2. It exists where $\vec{J}(\vec{r})=0$

3. It satisfies Laplace's equation

4. It is directly defined as $\Phi_m = -\int_a^b \vec{B} \cdot \vec{dl}$

5. It has the unit of Ampere.

(2) Magnetic vector Potential:

For the electric field case, we had seen that it is possible to define a scalar function Φ, called potential whose negative gradient is equal to the electric field $\vec{E} = -\vec{\nabla}\Phi$. The existence of such function is a consequence of the conservative nature of the electric force. It also followed that the electric field is ir-rotational, i.e $\vec{\nabla}\times\vec{E}=0$.

For magnetic field, Ampere's law gives a non zero curl, i.e $\vec{\nabla} \times \vec{B}(\vec{r}) = \mu_0 \vec{J}(\vec{r})$. Therefore we can not express $\vec{B}$ as a negative gradient of a scalar function as it would then violate Ampere's law. However, we may introduce a vector function $\vec{A}(\vec{r})$ such that $\vec{B} = \vec{\nabla} \times \vec{A}$. This would automatically satisfy $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{A}$ is called magnetic vector potential.

We know that a vector field uniquely determine by satisfying its divergence and curl. As $\vec{B}$ is a physical quantity, curl of $\vec{A}$ also so. However, $\vec{\nabla} \cdot \vec{A}$ has no physical meaning and consequently we are at liberate to specify its divergence as per our wish.

This freedom to choose a vector potential whose curl is $\vec{B}$ and whose divergence can be consequently chosen is called by mathematician as a choice of a **gauge.** If ψ is a scalar function any transformation of the type $\vec{A} \rightarrow \vec{A} + \vec{\nabla}\psi$ gives the same magnetic field as curl of gradient is identically zero. The transformation above is known as **gauge invariance** (we have a similar freedom for the scalar potential Φ of the electric field in the sense that it is determined up to an additive constant. Our most common choice of Φ is one for which $\Phi \rightarrow 0$ at infinity).

A popular **gauge** choice for $\vec{A}$ is one in which

$\vec{\nabla} . \vec{A} = 0 \rightarrow$ known as the condition of **coulomb gauge.**

<u>Biot-Savart law for vector potential:</u>

Biot-Savart law for magnetic field due to a current element $\vec{dl}$ is

$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, \vec{dl} \times \hat{r}}{r^2} = -\frac{\mu_0 I}{4\pi} \vec{dl} \times \vec{\nabla}(\frac{1}{r})$, where μ_0 is the permeability of free space.

Since the current element $\vec{dl}$ does not depend on the position vector of the point at which the magnetic field is calculated. We write, $d\vec{B} = \frac{\mu_0 I}{4\pi} \vec{\nabla} \times (\frac{\vec{dl}}{r})$. Thus the contribution to the vector potential from the current element $\vec{dl}$ is

$d\vec{A} = \frac{\mu_0 I}{4\pi} \frac{\vec{dl}}{r} \rightarrow \vec{A} = \frac{\mu_0 I}{4\pi} \int \frac{\vec{dl}}{r}$.

<u>The magnetic vector potential for different current configuration: (Unit-wb/m)</u>

<u>(1) Line current:</u>

$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I\, \vec{dl}}{r}$. I= ampere, $\vec{dl}$ =meter

<u>(2) Surface current:</u>

$\vec{A} = \frac{\mu_0}{4\pi} \iint \frac{\vec{K}\, ds}{r}$. $\vec{K} = \frac{I}{b}$ (Ampere/meter), ds= meter^2

<u>(3) Volume current:</u>

$\vec{A} = \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}\, dv}{r}$. $\vec{J} = \frac{I}{s}$ (Ampere/m^2), dv= meter^3

<u>Question:</u>

Show that magnetic vector potential $\vec{A}$ satisfied the Poisson's equation?

The set of equations which uniquely define the vector potential $\vec{A}$ and also satisfy the fundamental equation of Gauss's law $\vec{\nabla} . \vec{B} = 0$ are as follows

$\vec{B} = \vec{\nabla} \times \vec{A}$ ·········· (1) and $\vec{\nabla} . \vec{A} = 0$ ·········· (2)

From Ampere's law $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

$\rightarrow \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \mu_0 \vec{J} \rightarrow \vec{\nabla}(\vec{\nabla} . \vec{A}) - \nabla^2\vec{A} = \mu_0 \vec{J} \rightarrow - \nabla^2\vec{A} = \mu_0 \vec{J}$ [Using equation (1) and (2)]

$\nabla^2\vec{A} = - \mu_0 \vec{J} \rightarrow$ This equation is similar to Poisson's equation, the only difference is that $\vec{A}$ is a vector.

Therefore we can say that magnetic vector potential satisfy the Poisson's equation.

Characteristics of magnetic vector potential ($\vec{A}$):

1. It exists even when $\vec{J}$ is present.

2. It is defined in two ways $\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{A} = \frac{\mu_0}{4\pi} \iiint \frac{\vec{J}\, dv}{r}$. $\vec{J} = \frac{I}{s}$ (Ampere/m^2), dv= $meter^3$

3. $\nabla^2\vec{A} = - \mu_0 \vec{J}$.

4. $\nabla^2\vec{A} = 0$ if $\vec{J} = 0$.

5. Vector magnetic potential, $\vec{A}$ has applications to obtain radiation characteristics of antennas, apertures and also to obtain radiation leakage from transmission lines, waveguides and microwave ovens.

6. $\vec{A}$ is used to find near and far-fields of antennas.

Problems:

The vector magnetic potential, $\vec{A}$ due to a direct current in a conductor in free space is given by $\vec{A} = (x^2+y^2)\, \widehat{a_z}$ μwb/m. Determine the magnetic field produced by the current element at (1,2,3).

Ans. Given $\vec{A} = (x^2+y^2)\, \widehat{a_z}$ μwb/m and we know $\vec{B} = \vec{\nabla} \times \vec{A}$. Then

$$\vec{B} = 10^{-6} \begin{vmatrix} \widehat{a_x} & \widehat{a_y} & \widehat{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & (x2 + y2) \end{vmatrix}$$

$$\vec{B} = 10^{-6} \left[\frac{\partial}{\partial y}(x2 + y2)\widehat{a_x} - \frac{\partial}{\partial x}(x2 + y2)\widehat{a_y}\right]$$

$$\vec{B} = 10^{-6} \left[(x^2 + 2y)\widehat{a_x} - (2x + y^2)\widehat{a_y}\right]$$

$$(\vec{B})_{(1,2,3)} = 10^{-6} (5\widehat{a_x} - 6\widehat{a_y})$$

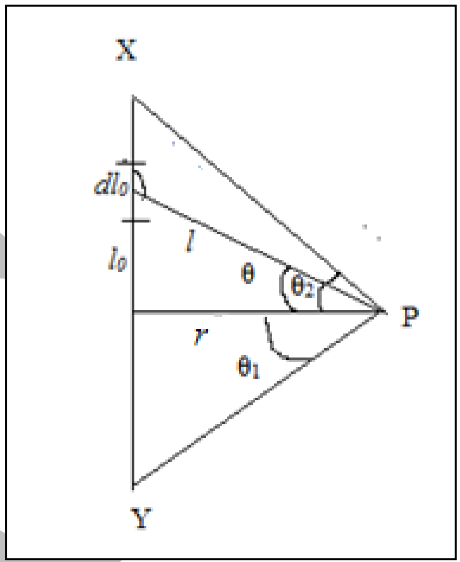
$$\vec{H} = \frac{1}{\mu_0} (5\widehat{a_x} - 6\widehat{a_y}) \times 10^{-6}$$

$$\vec{H} = \frac{1}{4\pi\times10^{-7}} (5\widehat{a_x} - 6\widehat{a_y}) \times 10^{-6}$$

$$\vec{H} = (3.978\, \widehat{a_x} - 4.774\widehat{a_y})$$

2. Determine the magnetic vector potential at a distance r from a very long thin straight wire carrying a current *I*. Hence find the corresponding magnetic field $\overrightarrow{B}$.

Ans. Let XY be a straight-line conductor carrying a current of *I* amp and P be a point at a distance r from XY. The magnetic vector potential $\vec{A}$ at P is required.



The magnetic vector potential at P due to element dl_0 of the wire is $d\vec{A} = \frac{\mu_0}{4\pi}\frac{I\,\overrightarrow{dl0}}{l} = \frac{\mu_0 I}{4\pi}\frac{r\,sec^2\theta}{r\sec\theta}\,d\theta\,\widehat{a_z}$

(From figure we can write $dl_0 = r\,sec^2\theta\,d\theta$, also $l = r\sec\theta$)

The magnetic vector potential at P due to entire current configuration is

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int_{-\theta 1}^{\theta 2}\sec\theta\,d\theta\,\widehat{a_z}$$

As the length L of the wire is very large compared with r, we have

$\theta 1 = \theta 2$. So, $\vec{A} = \frac{\mu_0 I}{2\pi}\int_0^{\theta 1}\sec\theta\,d\theta\,\widehat{a_z} = \frac{\mu_0 I}{2\pi}\ln(sec\theta_1 + tan\theta_1)$.

Since $L \gg r$, we have $sec\theta_1 \cong tan\theta_1 = \frac{L}{2r}$. Hence,

$\vec{A} = \frac{\mu_0 I}{2\pi}\ln(\frac{L}{2r})\,\widehat{a_z}$.

Using cylindrical coordinates (r,θ,z), we note that

$A_z = \frac{\mu_0 I}{2\pi}\ln(\frac{L}{2r})$, $A_r = 0$ and $A_\theta = 0$

Hence, $\vec{B} = \vec{\nabla}\times\vec{A} = -\,\widehat{a_\theta}\,\frac{d\,Az}{dr} = \widehat{a_\theta}\,\frac{\mu_0 I}{2\pi r}$, where $\widehat{a_\theta}$ is unit vector in the θ-direction.