# ELECTROSTATICS

## 2nd Semester Physics (Hons.) 2020

*Sagar Kumar Dutta*

Berhampore Girls' College
Govt. Sponsored

# Acknowledgement

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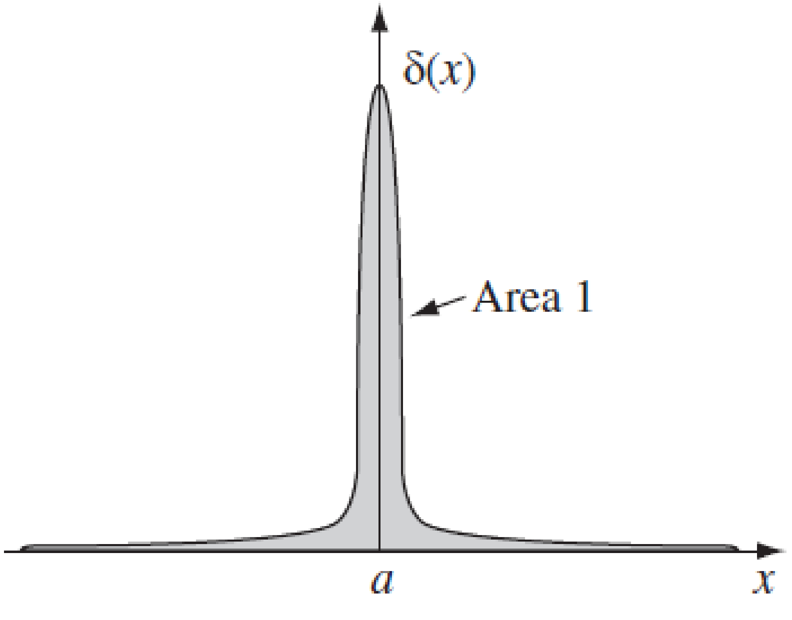
- Electricity & Magnetism – 8.02x (MITx)  Peter Dourmashkin

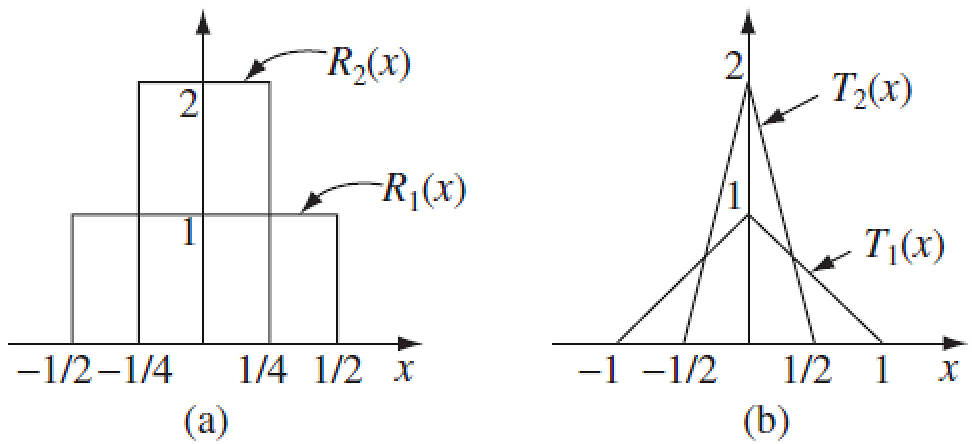
# 1-D Dirac Delta Function

$$\delta(x) = \left\{ \begin{array}{ll} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{array} \right\}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$



# 1-D Dirac Delta Function




Technically, $\delta(x)$ is not a function at all, since its value is not finite at $x = 0$; in the mathematical literature it is known as a **generalized function**, or **distribution**. It is, if you like, the *limit* of a *sequence* of functions, such as rectangles $R_n(x)$, of height $n$ and width $1/n$, or isosceles triangles $T_n(x)$, of height $n$ and base $2/n$

# 3-D Dirac Delta Function

$$\delta^3(\mathbf{r}) = \delta(x)\,\delta(y)\,\delta(z).$$

$$\int_{\text{all space}} \delta^3(\mathbf{r})\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(z)\,dx\,dy\,dz = 1.$$

$$\int_{\text{all space}} f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{a})\,d\tau = f(\mathbf{a}).$$

$$\boxed{\nabla \cdot \left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}}),}$$

$\boldsymbol{\mathscr{r}}$ is the separation vector: $\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}'$.

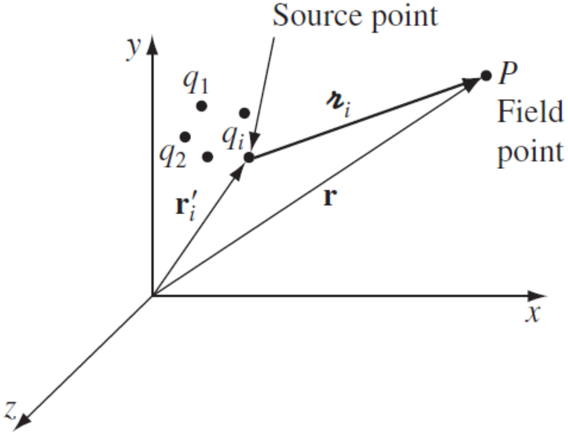
$$\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \qquad \nabla^2\frac{1}{\mathscr{r}} = -4\pi\delta^3(\boldsymbol{\mathscr{r}}).$$

# Coulomb's Law

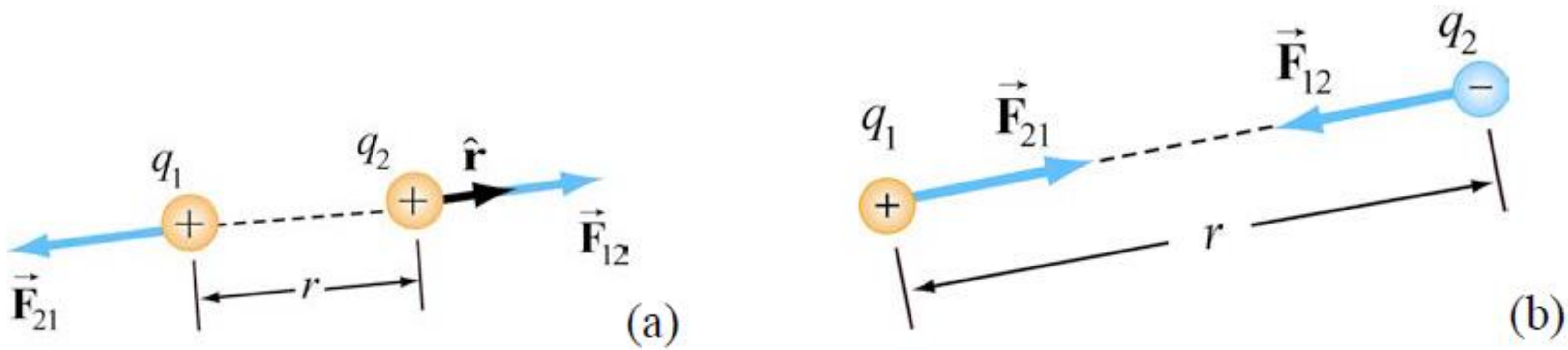
$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}.$$

Unit separation vector

**permittivity of free space** $\epsilon_0 = 8.85 \times 10^{-12} \dfrac{\mathrm{C}^2}{\mathrm{N} \cdot \mathrm{m}^2}.$



# Coulomb's Law




$$k_e = \frac{1}{4\pi\varepsilon_0} = 8.9875\times10^9\,\mathrm{N\cdot m^2/C^2} \qquad \varepsilon_0 = \frac{1}{\mu_0 c^2},$$

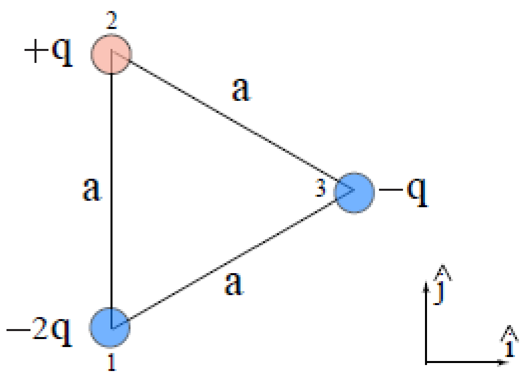
where the constant $\mu_0 = 4\pi\times10^{-9}\ \mathrm{N\cdot s^2\cdot C^{-2}}$ is called the *permeability of free space*, and $c = 299792458\,\mathrm{m\cdot s^{-1}}$ is the speed of light. Therefore

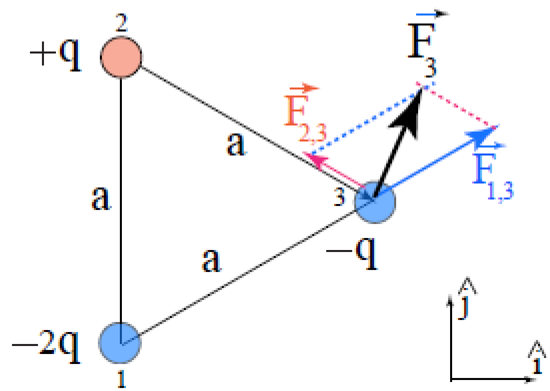
$$\varepsilon_0 = \frac{1}{\mu_0 c^2} = \frac{1}{(4\pi\times10^{-9}\ \mathrm{N\cdot s^2\cdot C^{-2}})(299792458\,\mathrm{m\cdot s^{-1}})^2}$$
$$= 8.854187817\ldots\times10^{-12}\ \mathrm{C^2/N\cdot m^2}.$$

# Principle of Superposition

Coulomb's law applies to any pair of point charges. When more than two charges are present, the net force on any one charge is simply the vector sum of the forces exerted on it by the other charges. For example, if three charges are present, the resultant force experienced by $q_3$ due to $q_1$ and $q_2$ will be

$$\vec{\mathbf{F}}_3 = \vec{\mathbf{F}}_{13} + \vec{\mathbf{F}}_{23}\,.$$

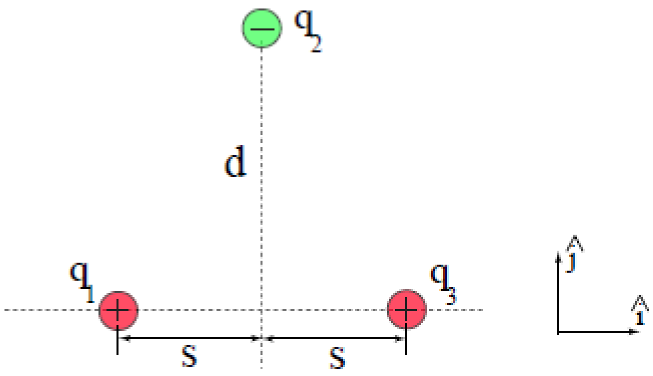



$$\vec{F}_3 = \vec{F}_{1,3} + \vec{F}_{2,3}$$

$$= k\frac{q^2}{2a^2}(\sqrt{3}\,\hat{\imath} + 3\,\hat{\jmath})$$

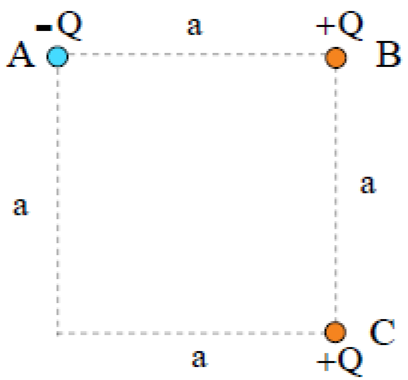
# Total Force on a charged object



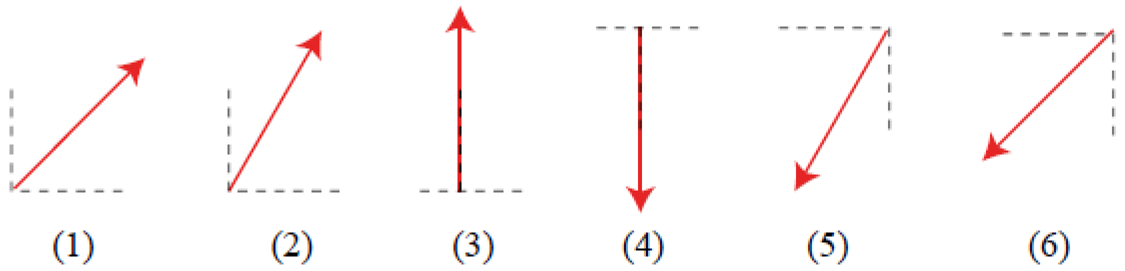

Assume $q_1 = q_3 = +Q$ and $q_2 = -Q$, where $Q > 0$.

Calculate, $\vec{F}_2$, the force exerted by objects 1 and 3 on object 2.

# Direction of the Total Force

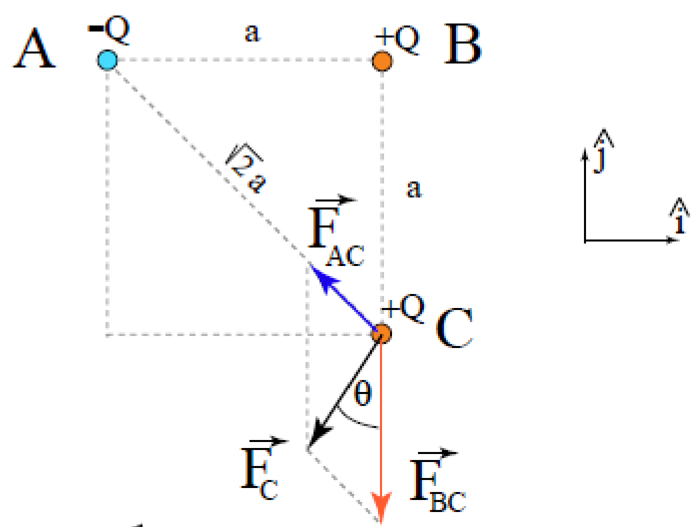



Which of the arrows below best represents the approximate direction of the total force on the charge at point $C$?




(7) None of these.

# Direction of the Total Force




$$|\vec{F}_{AC}| \sim \frac{1}{2a^2} \qquad |\vec{F}_{BC}| \sim \frac{1}{a^2}$$

$$\theta = \tan^{-1}\left(\frac{|F_x|}{|F_y|}\right)$$

$$\vec{F}_C = \vec{F}_{A,C} + \vec{F}_{B,C} = k\frac{Q^2}{2a^2}\left(-\frac{\sqrt{2}}{2}\hat{\mathbf{i}} + \frac{\sqrt{2}}{2}\hat{\mathbf{j}}\right) - k\frac{Q}{a^2}\hat{\mathbf{j}}$$

$$= \tan^{-1}\left(\frac{\sqrt{2}/4}{1-\sqrt{2}/4}\right)$$

$$\vec{F}_C = k\frac{Q^2}{a^2}\left(-\frac{\sqrt{2}}{4}\hat{\mathbf{i}} + \left(\frac{\sqrt{2}}{4} - 1\right)\hat{\mathbf{j}}\right)$$

$$= 28.7^\circ < 45^\circ$$

# The Electric Field

- $$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{q_2 Q}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + \ldots\right)$$

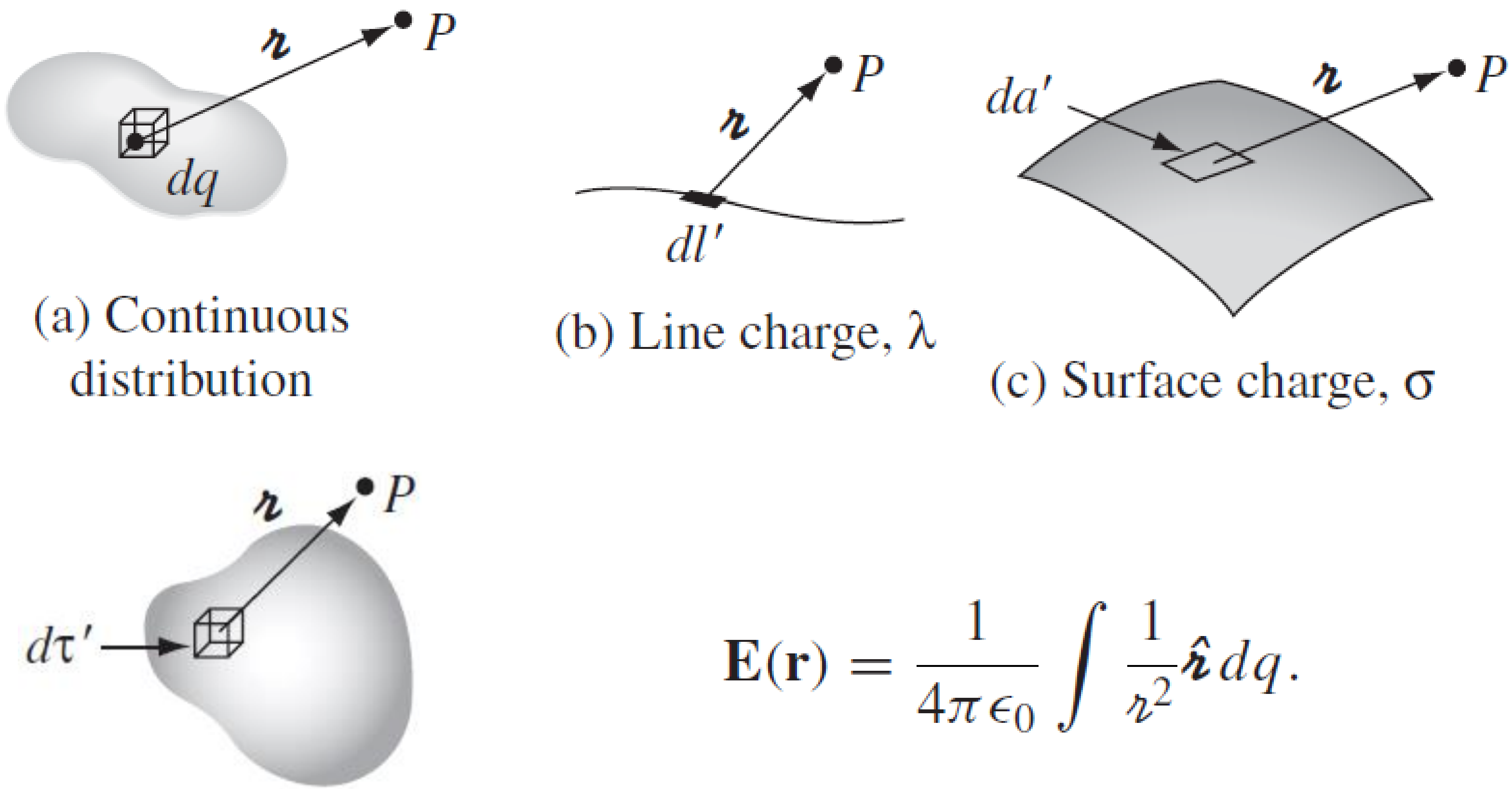
$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{r}_1^2}\hat{\boldsymbol{\mathscr{r}}}_1 + \frac{q_2}{\mathscr{r}_2^2}\hat{\boldsymbol{\mathscr{r}}}_2 + \frac{q_3}{\mathscr{r}_3^2}\hat{\boldsymbol{\mathscr{r}}}_3 + \ldots\right),$$

$$\boxed{\mathbf{F} = Q\mathbf{E},}$$

Electric field for discrete point charges

$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\boldsymbol{\mathscr{r}}}_i.$$

# Continuous Charge Distributions




(a) Continuous distribution
(b) Line charge, λ
(c) Surface charge, σ
(d) Volume charge, ρ

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,dq.$$

# Electric field for continuous charge distributions

$$dq \rightarrow \lambda\, dl' \sim \sigma\, da' \sim \rho\, d\tau'$$

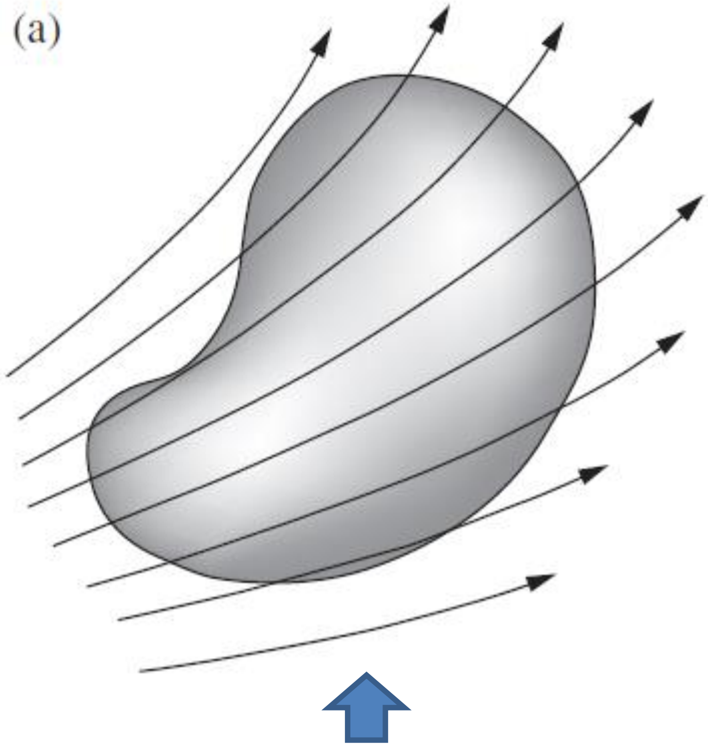
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, dl'$$ ➡ Line charge

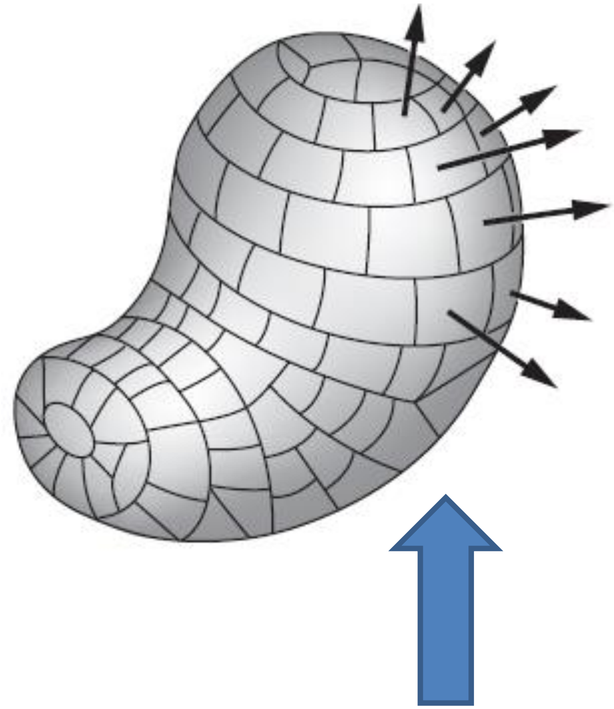
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, da'$$ ➡ Surface charge

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, d\tau'$$ ➡ Volume charge

# *Gauss's Law*

(a)



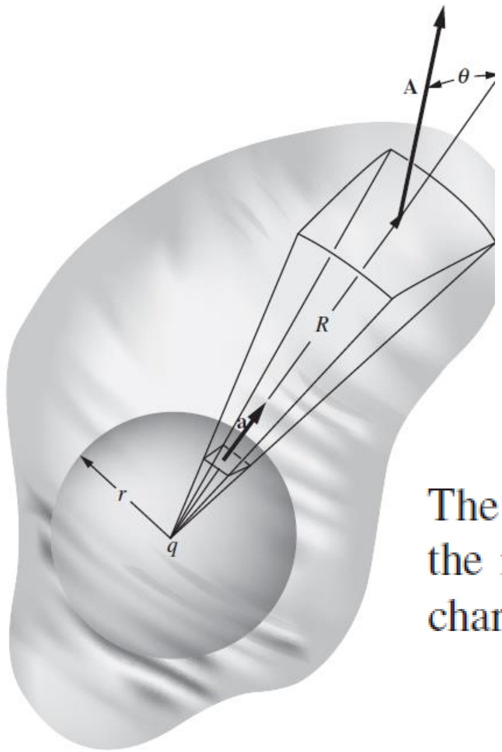


Flux

Unit vectors of area elements

# Gauss's Law

Flux $\Phi = \int_S \mathbf{E} \cdot d\mathbf{a} = \int_S (\mathbf{E}_1 + \mathbf{E}_2 + \cdots + \mathbf{E}_N) \cdot d\mathbf{a}.$




$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \sum_i q_i = \frac{1}{\epsilon_0} \int \rho \, dv \qquad \text{(Gauss's law)}$$

The flux of the electric field $\mathbf{E}$ through any closed surface, that is, the integral $\int \mathbf{E} \cdot d\mathbf{a}$ over the surface, equals $1/\epsilon_0$ times the total charge enclosed by the surface:

# *Gauss's law in differential form*

- By applying divergence theorem

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau$$

$$Q_{\text{enc}} = \int_{\mathcal{V}} \rho\, d\tau$$

So Gauss's law becomes

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0}\right) d\tau$$

And since this holds for *any* volume, the integrands must be equal:

$$\boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho.}$$

# The Divergence of E

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \rho(\mathbf{r}')\, d\tau'$$

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \nabla \cdot \left( \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \rho(\mathbf{r}')\, d\tau'$$

$$\nabla \cdot \left( \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = 4\pi \delta^3(\boldsymbol{\mathscr{r}})$$

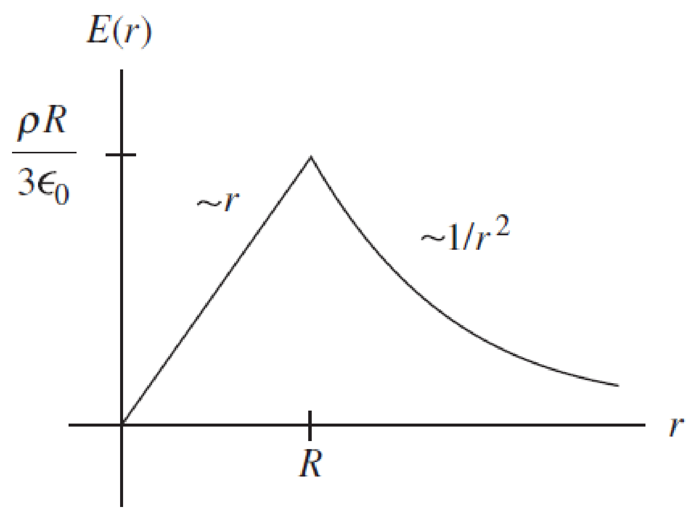
$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int 4\pi \delta^3(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}')\, d\tau' = \frac{1}{\epsilon_0} \rho(\mathbf{r})$$

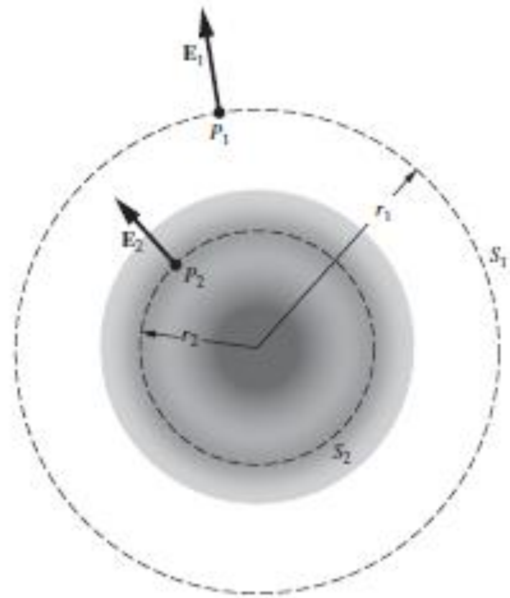
$$\int_{\mathcal{V}} \nabla \cdot \mathbf{E}\, d\tau = \oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int_{\mathcal{V}} \rho\, d\tau = \frac{1}{\epsilon_0} Q_{\text{enc}}.$$

# Field inside and outside a uniform sphere

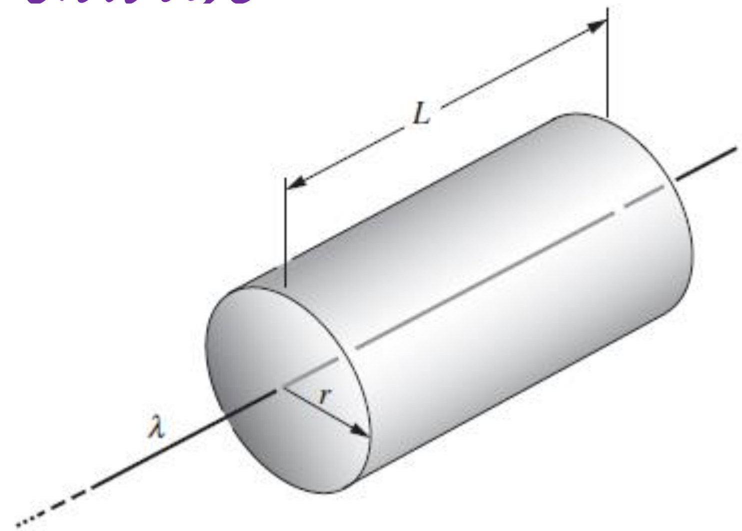
$$E(r) = \frac{(4\pi R^3/3)\rho}{4\pi\epsilon_0 r^2} = \frac{\rho R^3}{3\epsilon_0 r^2} \qquad (r \geq R)$$

$$E(r) = \frac{(4\pi r^3/3)\rho}{4\pi\epsilon_0 r^2} = \frac{\rho r}{3\epsilon_0} \qquad (r \leq R)$$



# *Field of a line charge*




The flux through the cylindrical surface is simply the area, $2\pi rL$, times $E_r$, the field at the surface. On the other hand, the charge enclosed by the surface is just $\lambda L$, so Gauss's law gives us $(2\pi rL)E_r = \lambda L/\epsilon_0$ or

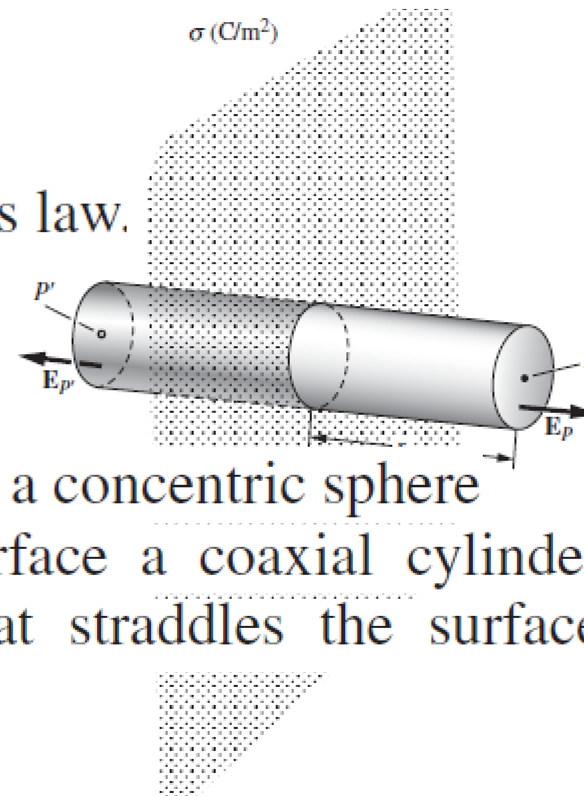
$$E_r = \frac{\lambda}{2\pi\epsilon_0 r}$$

# *Field of an infinite sheet of charge*

outward flux is found only at the ends, so that if $E_P$ denotes the magnitude of the field at $P$, and $E_{P'}$ the magnitude at $P'$, the outward flux is $AE_P + AE_{P'} = 2AE_P$. The charge enclosed is $\sigma A$, so Gauss's law gives $2AE_P = \sigma A/\epsilon_0$, or

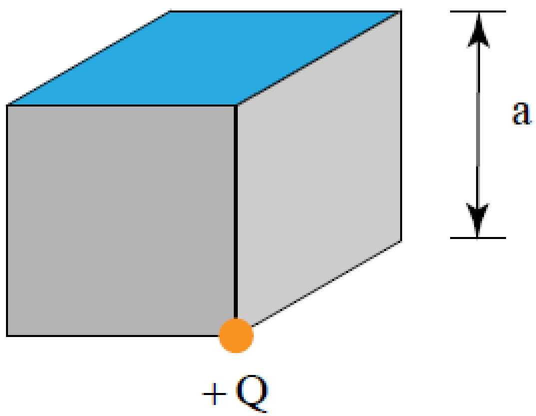
$$E_P = \frac{\sigma}{2\epsilon_0}$$

*Symmetry is crucial* to this application of Gauss's law.




1. *Spherical symmetry.* Make your Gaussian surface a concentric sphere
2. *Cylindrical symmetry.* Make your Gaussian surface a coaxial cylinder
3. *Plane symmetry.* Use a Gaussian "pillbox" that straddles the surface

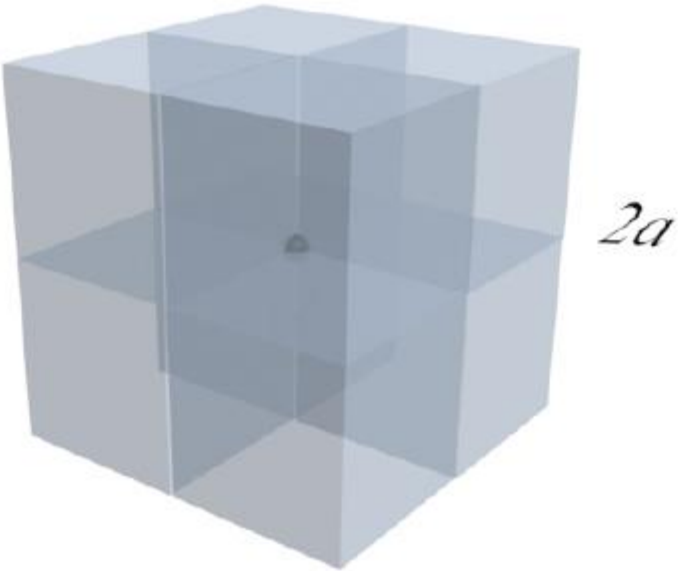
# Flux in a Cube (Problem)




Consider a cube with each side of length $a$. A point-like object with charge $Q$ is placed at one corner of the cube which is shared by three faces, as show in the sketch. What is the flux of the electric field emerging from each of the other three square faces of the cube, that is, the three faces which do not have the charge at one of their corners (one of these faces is show in blue at the top of the image above)?

# Flux in a Cube (Solution)

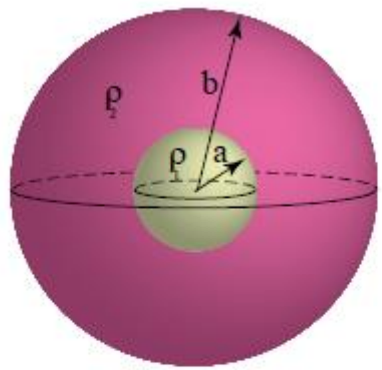
Consider a cube of side $2a$. This cube can be constructed from eight cubes each of side $a$.




Place a point-like object with charge $Q$ at the center of this larger cube. The total flux through the larger cube, by Gauss's law, is just $Q/\varepsilon_0$. Since each face of the larger cube is identical (i.e. the same distance away from the point charge and with the same orientation), the flux through each of those six larger faces is 1/6 of the total flux or $Q/6\varepsilon_0$. We can divide each larger face into four equal square faces of side $a$ which again have identical distances from the charge and orientations. Therefore, the flux through each of these smaller square faces is $Q/24\varepsilon_0$. The flux though one of these smaller square faces is the same as the flux through each face of a cube of side $a$ with the charged object placed at the corner, the quantity we would like to determine. So the answer is

$$\Phi_E = \frac{Q}{24\varepsilon_0}.$$

# *Sphere with Non-Uniform charge distribution*




A non-conducting sphere of radius $b$ is constructed of two materials. The inner portion, radius $a$, has a non-uniform volume charge density given by:

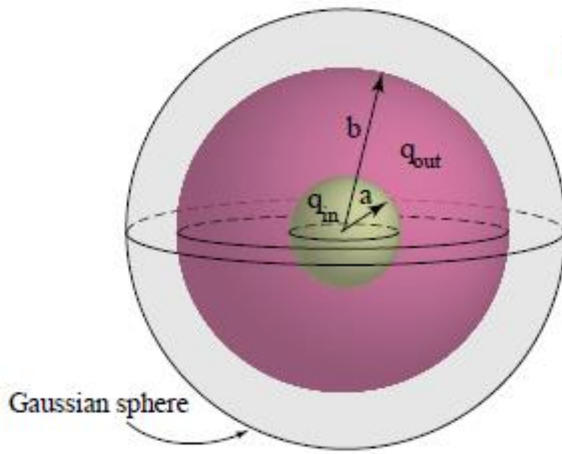
$$\rho_1(r) = \frac{\alpha}{r} \quad \text{for} \quad r < a$$

where $\alpha > 0$ is a constant. The outer portion, with inner radius $a$ and outer radius $b$ has a uniform charge density $\rho_2$.

**(Part a)** Calculate $q_{in}$, the total charge in the inner region of radius $a$.

**(Part b)** If the electric field outside the sphere is everywhere zero, $\vec{\mathbf{E}} = 0$ for $r > b$, what is the uniform charge density $\rho_2$ of the outer portion of the sphere?

# Sphere with Non-Uniform charge distribution




$$\vec{E} = 0 \quad r > b$$

$$q_{enc} = 0 \Longrightarrow q_{out} = -q_{in}$$

$$\begin{aligned} q_{in} &= \int_{\text{inner region}} \rho_1 dV \\ &= \int_{r'=0}^{r'=a} \rho_1(r') 4\pi r'^2 dr' \\ &= \int_{r'=0}^{r'=a} \frac{\alpha}{r'} 4\pi r'^2 dr' \\ q_{in} &= 2\alpha\pi a^2 \end{aligned}$$

$$\begin{aligned} q_{out} &= \int_{r'=a}^{r'=b} \rho_2 dV \\ &= \int_{r'=a}^{r'=b} \rho_2 4\pi r'^2 dr' \\ &= \frac{\rho_2 4\pi(b^3 - a^3)}{3} \end{aligned}$$

Set $q_{out} = -q_{in}$ and use the value of $q_{in} = 2\alpha\pi a^2$

$$\rho_2 = \frac{3q_{out}}{4\pi(b^3 - a^3)} \quad \Rightarrow \quad \rho_2(r) = -\frac{3\alpha a^2}{2(b^3 - a^3)} \text{for} \quad a < r < b$$

# Sphere with Non-Uniform charge distribution

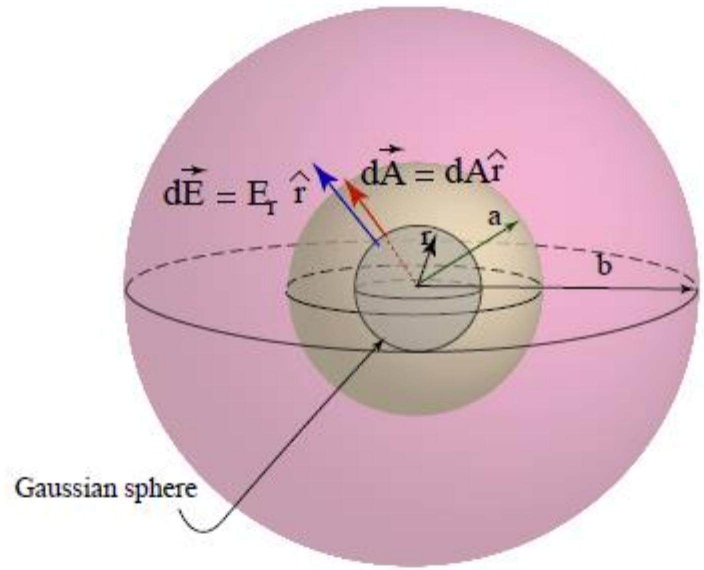
**(Part c)** Calculate the magnitude of the radial component of the electric field at any point inside the inner region of the sphere ($r < a$).

Due to the spherical symmetry $\vec{\mathbf{E}} = E_r(r)\hat{\mathbf{r}}$

$$q_{enc} = \int_{\text{Gaussian sphere}} \rho_1 dV$$

$$= \int_{r'=0}^{r'=r} \rho_1(r') 4\pi r'^2 dr'$$

$$= \int_{r'=0}^{r'=r} \frac{\alpha}{r'} 4\pi r'^2 dr'$$




$$q_{enc} = 2\alpha\pi r^2$$

$$\oiint_{\text{Gaussian sphere}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oiint_{\text{Gaussian sphere}} E_r dA = E_r \oiint_{\text{Gaussian sphere}} dA = E_r 4\pi r^2$$

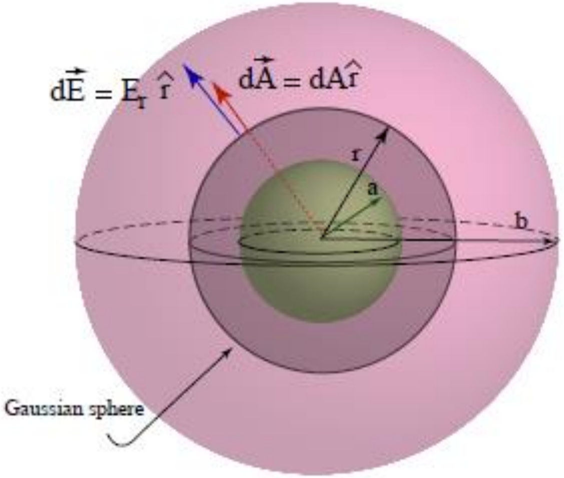
$$\oiint_{\text{Gaussian sphere}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{enc}}{\varepsilon_0} \Rightarrow E_r 4\pi r^2 = \frac{\alpha 2\pi r^2}{\varepsilon_0} \Rightarrow E_r = \frac{\alpha}{2\varepsilon_0}$$

$$\boxed{\vec{\mathbf{E}} = \frac{\alpha}{2\varepsilon_0}\hat{\mathbf{r}}}$$

# Sphere with Non-Uniform charge distribution

**(Part d)** Calculate $E_r$, the magnitude of the radial component of the electric field at any point in the outer region of the sphere, $a < r < b$.

$$
\begin{aligned}
q_{enc} &= q_{in} + \int_{r'=a}^{r'=r} \rho_2 4\pi r'^2 \, dr' \\
&= 2\alpha\pi a^2 - \frac{3\alpha a^2}{2(b^3 - a^3)} \frac{4\pi}{3}(r^3 - a^3) \\
&= 2\alpha\pi a^2 \left(1 - \frac{r^3 - a^3}{b^3 - a^3}\right)
\end{aligned}
$$



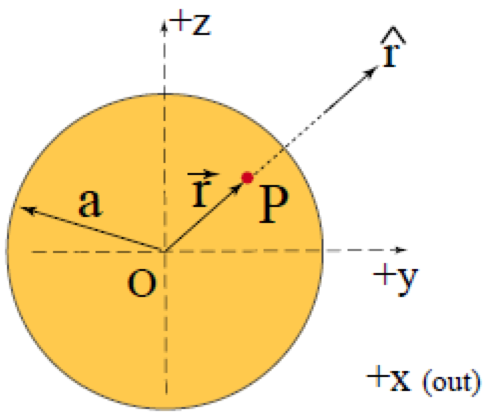

the electric flux through the Gaussian sphere of radius $r$ is $E_r \cdot 4\pi r^2$, therefore Gauss's law becomes:

$$E_r \cdot 4\pi r^2 = \frac{2\alpha\pi a^2}{\varepsilon_0}\left(1 - \frac{r^3 - a^3}{b^3 - a^3}\right) \Rightarrow \boxed{\vec{\mathbf{E}} = \frac{\alpha a^2}{2r^2\varepsilon_0}\left(1 - \frac{r^3 - a^3}{b^3 - a^3}\right)\hat{\mathbf{r}};\ a < r < b.}$$

At $r = b$ the electric field is zero as expected.

At $r = a$ the electric field is $\vec{\mathbf{E}} = \dfrac{\alpha}{2\varepsilon_0}\hat{\mathbf{r}}$

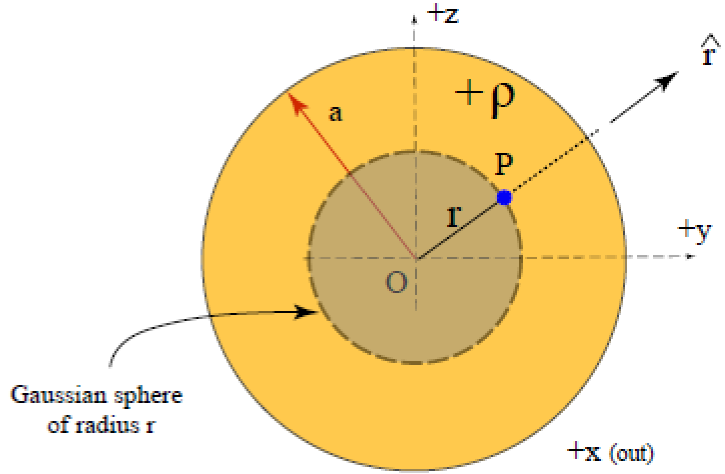
# Non-Conducting Solid Sphere (Part a)




A non conductive sphere of radius $a$, has a uniform charge distribution $\rho$. The center of the sphere is at the origin of the coordinate system.

Calculate the electric field at a point $P$ inside the sphere, where the position of point $P$ is given as $\vec{r} = r\hat{r}$.

# Non-Conducting Solid Sphere (Part a)




Consider the point $P$ inside the sphere, a distance $r$ from the origin. We choose a sphere of radius $r$ as our Gaussian surface with $r < a$.

The electric field at any point inside the sphere is radially outwards and given by $\vec{\mathbf{E}} = E_1(r)\hat{\mathbf{r}}$, where $E_1(r)$ is the radial component of the electric field and is only a function of $r$. The electric flux through this closed surface is

$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E_1 \cdot 4\pi r^2$$

Because the charge distribution is uniform, the charge enclosed by the Gaussian surface is

$$\frac{Q_{\text{enc}}}{\varepsilon_0} = \frac{\rho\left(4\pi r^3 \,/\, 3\right)}{\varepsilon_0}$$

Recall that Gauss' Law equates electric flux to charge enclosed:

# Non-Conducting Solid Sphere

$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\text{enc}}}{\varepsilon_0}$$

So we substitute the two calculations above into Gauss' law to arrive at:

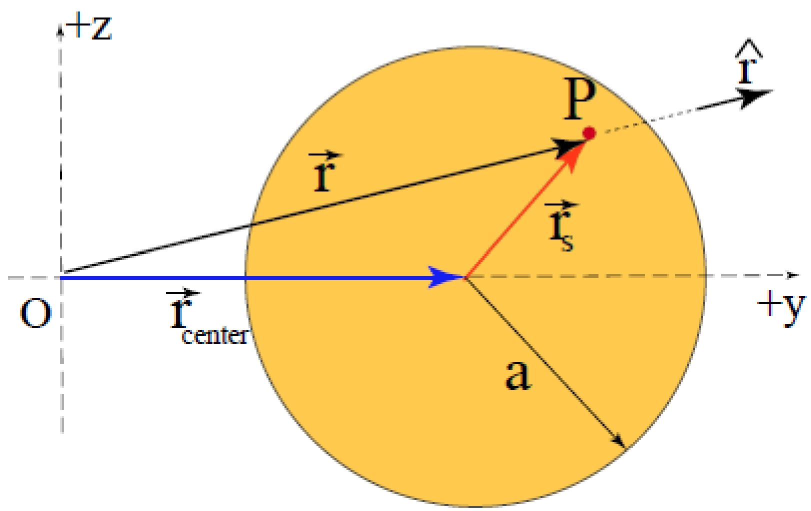
$$E_1 \cdot 4\pi r^2 = \frac{\rho \left(4\pi r^3 / 3\right)}{\varepsilon_0}$$

We can solve this equation for the electric field

$$\vec{\mathbf{E}}\left(P\right) = E_1 \hat{\mathbf{r}} = \frac{\rho r}{3\varepsilon_0} \hat{\mathbf{r}}$$

This answer can also be written as $\vec{\mathbf{E}}\left(P\right) = \frac{\rho}{3\varepsilon_0} r \hat{\mathbf{r}} = \frac{\rho}{3\varepsilon_0} \vec{\mathbf{r}}$.

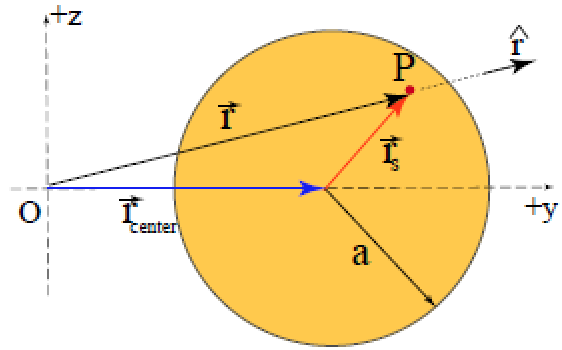
# Non-Conducting Solid Sphere (Part b)




**(Part b)** We translate the sphere along the y-axis so that its center is at point $\vec{r}_{\text{center}} = b\hat{j}$, where $b > a$. The position of point $P$ inside the sphere, measured with respect to the center of the sphere is $\vec{r}_s$, and the position of point $P$ measured from the coordinate system with origin at point $O$ is $\vec{r} = r\hat{r}$.

Write the expression of the electric field at point $P$ inside the sphere, with respect to the coordinate system of origin in $O$.

# Non-Conducting Solid Sphere (Part b)




**(Part b)** The electric field at point $P$ has the same magnitude as the one calculated in part (a) and it points radially outwards from the center of the sphere. In terms of the coordinate system with origin at the center of the sphere, the electric field at point $P$ is given by:

$$\vec{\mathbf{E}} = \frac{\rho}{3\varepsilon_0}\vec{r}_s \quad (1)$$

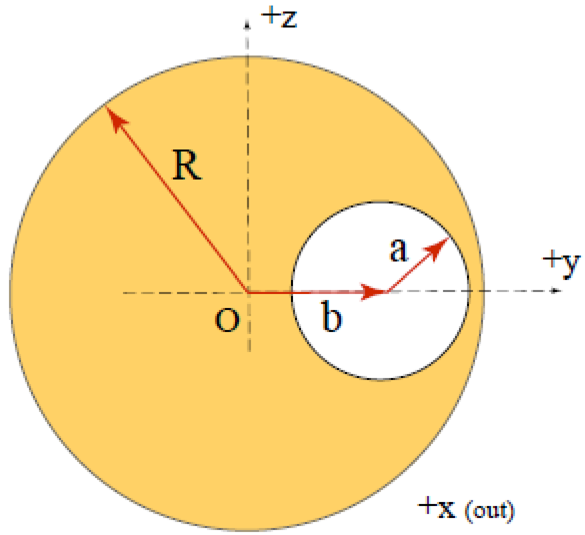
Looking at the figure, vector $\vec{r}$ is the sum of vectors $\vec{r}_s$ and $\vec{r}_{\text{center}}$:

$$\vec{r} = \vec{r}_s + \vec{r}_{\text{center}} \Longrightarrow \vec{r}_s = \vec{r} - \vec{r}_{\text{center}} = r\hat{\mathbf{r}} - b\hat{\mathbf{j}} \quad (2)$$

Replacing (eq. 2) in (eq. 1) we obtain:

$$\vec{\mathbf{E}} = \frac{\rho}{3\varepsilon_0}(r\hat{\mathbf{r}} - b\hat{\mathbf{j}}) \quad (3)$$

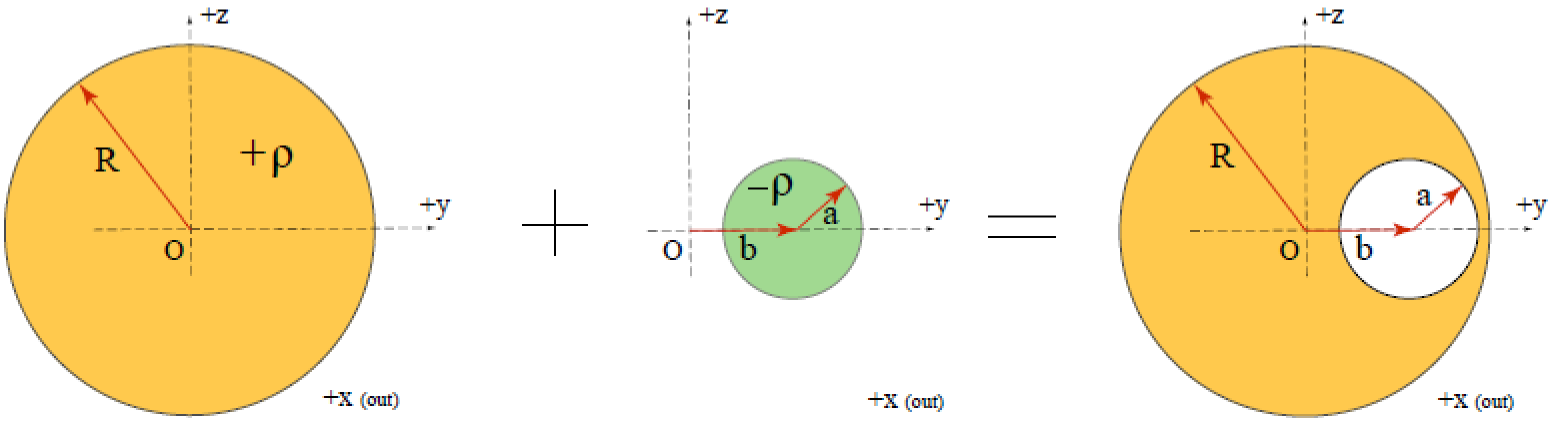
# Non-Conducting Solid Sphere (Part-c)




**(Part c)** Consider now a sphere of radius $R$ made of a non-conducting material that has a uniform volume charge density $\rho$. The center of the sphere is at the origin of the coordinate system. A spherical cavity of radius $a$ (with $a < R$) is then carved out from the sphere. As shown in the figure above, the center of the cavity is on the y-axis a distance $b$ from the origin. Find the direction and magnitude of the electric field at any point within the cavity.

# Non-Conducting Solid Sphere(Part-c)

At first glance this charge distribution does not seem to have any of the symmetries that enable us to use Gauss's law. However we can consider this charge distribution as the sum of two uniform spherical distributions of charge. The first is a sphere of radius $R$ centered at the origin with a uniform volume charge density $\rho$. The second is a sphere of radius $a$ centered at the point along the $y$-axis a distance $b$ from the origin (the center of the spherical cavity) with a uniform volume charge density $-\rho$.




When we add together these two distributions of charge we obtain the desired uniform charged sphere with a spherical cavity of radius $a$. We can then add together the electric fields from these two distributions at any point in the cavity to obtain the electric field of the original distribution at that point inside the cavity (superposition principle). Each of these two distributions are spherically symmetric and therefore we can use Gauss's Law to find the electric field associated with each of them.

# Non-Conducting Solid Sphere(Part-c)

**Electric field inside the sphere of radius a:**

In part (b), we calculated the electric field of a uniform charged sphere of radius $a$ with center at point $\vec{r}_{\text{center}}$. The sphere in part (b) was positively charged therefore we have to change $\rho$ to $-\rho$ in (eq. 3). We call this electric field $\vec{\mathbf{E}}_{-}$

$$\vec{\mathbf{E}}_{-} = -\frac{\rho}{3\varepsilon_0}(r\hat{\mathbf{r}} - b\hat{\mathbf{j}}) \qquad (4)$$

**Electric field inside the sphere of radius R:**

This electric field was found in part (a):

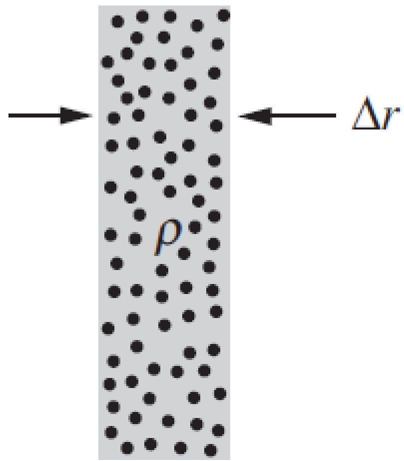
$$\vec{\mathbf{E}}_{+}(P) = \frac{\rho r}{3\varepsilon_0}\hat{\mathbf{r}} = \frac{\rho}{3\varepsilon_0} r\,\hat{\mathbf{r}} \qquad (5)$$

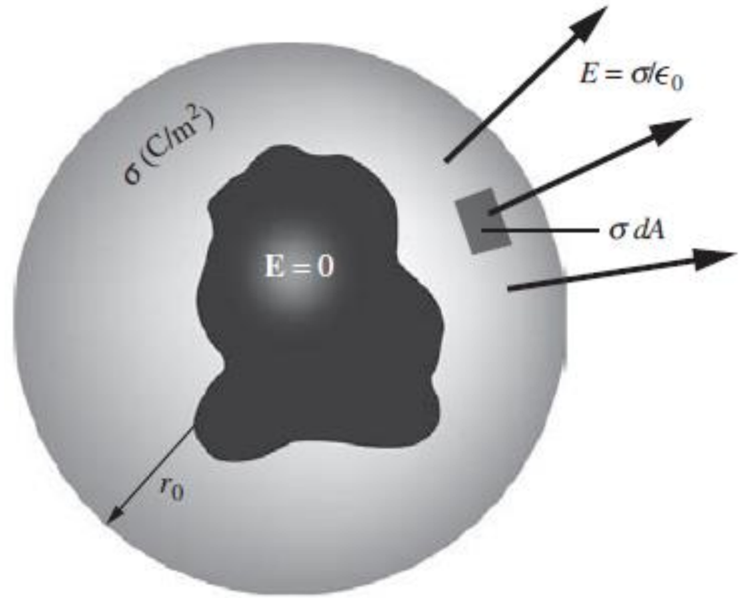
Now we use superposition principle with the expressions in (eq. 5) and (eq. 4) for $\vec{\mathbf{E}}_{+}$ and $\vec{\mathbf{E}}_{-}$ to obtain the electric field for the charge distribution of a sphere with a cavity:

$$\begin{aligned}
\vec{\mathbf{E}}(P) &= \vec{\mathbf{E}}_{+}(P) + \vec{\mathbf{E}}_{-}(P) \\
&= \frac{\rho}{3\varepsilon_0} r\hat{\mathbf{r}} - \frac{\rho}{3\varepsilon_0}(r\hat{\mathbf{r}} - b\hat{\mathbf{j}}) \\
&= \frac{\rho}{3\varepsilon_0} b\hat{\mathbf{j}}
\end{aligned}$$

# The force on a layer of charge

$$E_{\text{just outside}} = \frac{\sigma}{\epsilon_0}$$

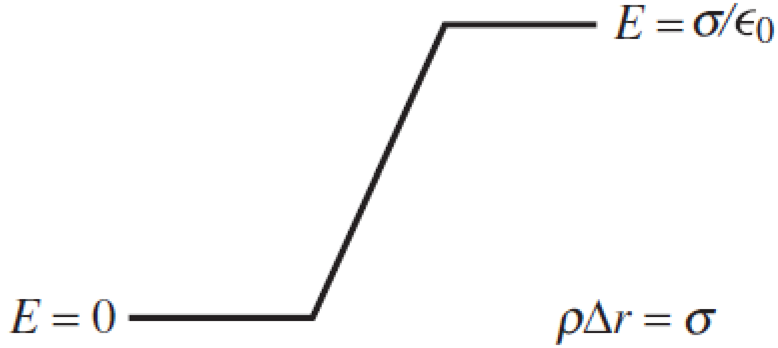



$$d\mathbf{F} = \mathbf{E}\, dq = \mathbf{E}\sigma\, dA$$

Newton's third law; the patch as a whole cannot push on itself. That simplifies our problem, for it allows us to use the entire electric field **E**, *including* the field due to all charges in the patch, in calculating the force $d\mathbf{F}$ on the patch of charge $dq$:

# The force on a layer of charge




But what $E$ shall we use, the field $E = \sigma/\epsilon_0$ outside the sphere or the field $E = 0$ inside? The correct answer, as we shall prove in a moment, is the *average* of the two fields that is,

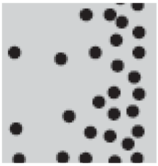
$$dF = \frac{1}{2}(\sigma/\epsilon_0 + 0)\sigma\, dA = \frac{\sigma^2\, dA}{2\epsilon_0}$$

# The force on a layer of charge

Now let us look carefully within the layer where the field is changing continuously from $E_1$ to $E_2$ and there is a volume charge density $\rho(x)$ extending from $x = 0$ to $x = x_0$, the thickness of the layer

Consider a much thinner slab, of thickness $dx \ll x_0$, which contains per unit area an amount of charge $\rho\, dx$. If the area of this thin slab is $A$, the force on it is

$$dF = E\rho\, dx \cdot A.$$



Thus the total force per unit area of our original charge layer is

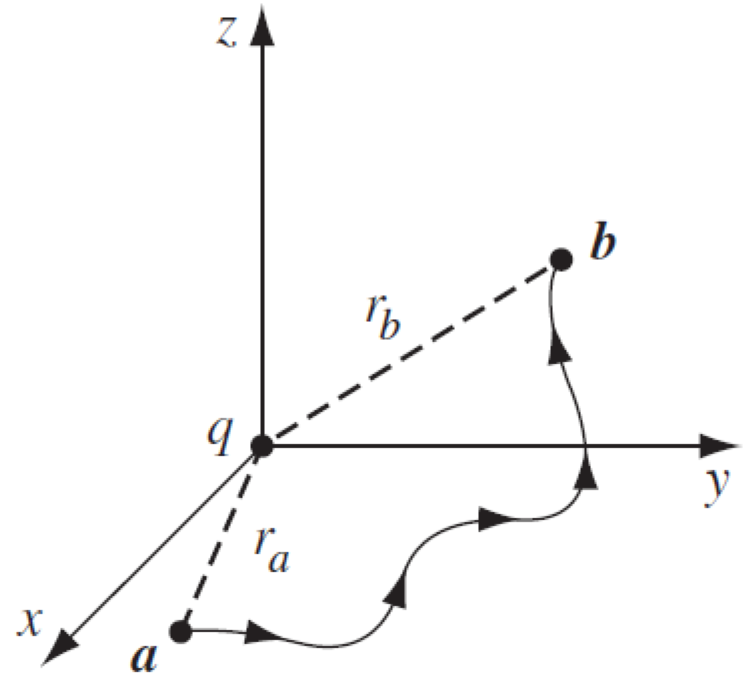
$$\frac{F}{A} = \int \frac{dF}{A} = \int_0^{x_0} E\rho\, dx \quad \Rightarrow \quad \frac{F}{A} = \int_{E_1}^{E_2} \epsilon_0 E\, dE = \frac{\epsilon_0}{2}\left(E_2^2 - E_1^2\right)$$

**Electrostatic Pressure**

# Electric Potential

For a point charge → $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$

$$\mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dr$$




In spherical coordinates, $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$.

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}\cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{a}}^{\mathbf{b}} \frac{q}{r^2}dr = \frac{-1}{4\pi\epsilon_0}\frac{q}{r}\Bigg|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

# Electric Potential

The integral around a *closed* path is evidently zero (for then $r_a = r_b$)

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0$$

applying Stokes' theorem, ⇒ $\nabla \times \mathbf{E} = \mathbf{0}$

$$\nabla \times \mathbf{E} = \frac{1}{4\pi\epsilon_0} \nabla \times \int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \rho \, d\tau = \frac{1}{4\pi\epsilon_0} \int \left[ \nabla \times \left( \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \right] \rho \, d\tau$$

$$= \mathbf{0}$$

$$\boxed{\nabla \times \left( \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) = \mathbf{0}}$$

# Electric Potential

$$V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}.$$

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_{\mathcal{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l}$$

$$= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathbf{a}}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

$$V(\mathbf{b}) - V(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l} \Rightarrow \int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$

$$\boxed{\mathbf{E} = -\nabla V.}$$

# Potential of a uniform charged sphere

$$V(r) = -\int_{\infty}^{r} \mathbf{E}\cdot d\mathbf{l} \begin{cases} \text{Outside the sphere } (r > R): \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} \\ \text{Inside the sphere } (r < R): \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}} \end{cases}$$

$$r > R\text{: } V(r) = -\int_{\infty}^{r}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{\bar{r}^2}\right)d\bar{r} = \frac{1}{4\pi\epsilon_0}q\left(\frac{1}{\bar{r}}\right)\Big|_{\infty}^{r} \Rightarrow \frac{q}{4\pi\epsilon_0}\frac{1}{r}$$

$$r < R\text{: } V(r) = -\int_{\infty}^{R}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{\bar{r}^2}\right)d\bar{r} - \int_{R}^{r}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\bar{r}\right)d\bar{r}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R} - \frac{1}{R^3}\left(\frac{r^2 - R^2}{2}\right)\right] \Rightarrow \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

$$r > R,\ \nabla V = \frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{\mathbf{r}} \Rightarrow = -\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$$

$$r < R,\ \nabla V = \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\frac{\partial}{\partial r}\left(3 - \frac{r^2}{R^2}\right)\hat{\mathbf{r}} \Rightarrow = \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(-\frac{2r}{R^2}\right)\hat{\mathbf{r}}$$

# Poisson's Equation and Laplace's Equation

$$\mathbf{E} = -\boldsymbol{\nabla} V$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{and} \qquad \boldsymbol{\nabla} \times \mathbf{E} = \mathbf{0}$$

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \boldsymbol{\nabla} \cdot (-\boldsymbol{\nabla} V) = -\nabla^2 V$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$ ⟹ **Poisson's Equation**

In regions where there is no charge, $\rho = 0$

Poisson's equation reduces to **Laplace's equation**

$$\nabla^2 V = 0.$$ ⟶ **Laplace's Equation**

# Poisson's Equation

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathcal{r}}\,d\tau'$$

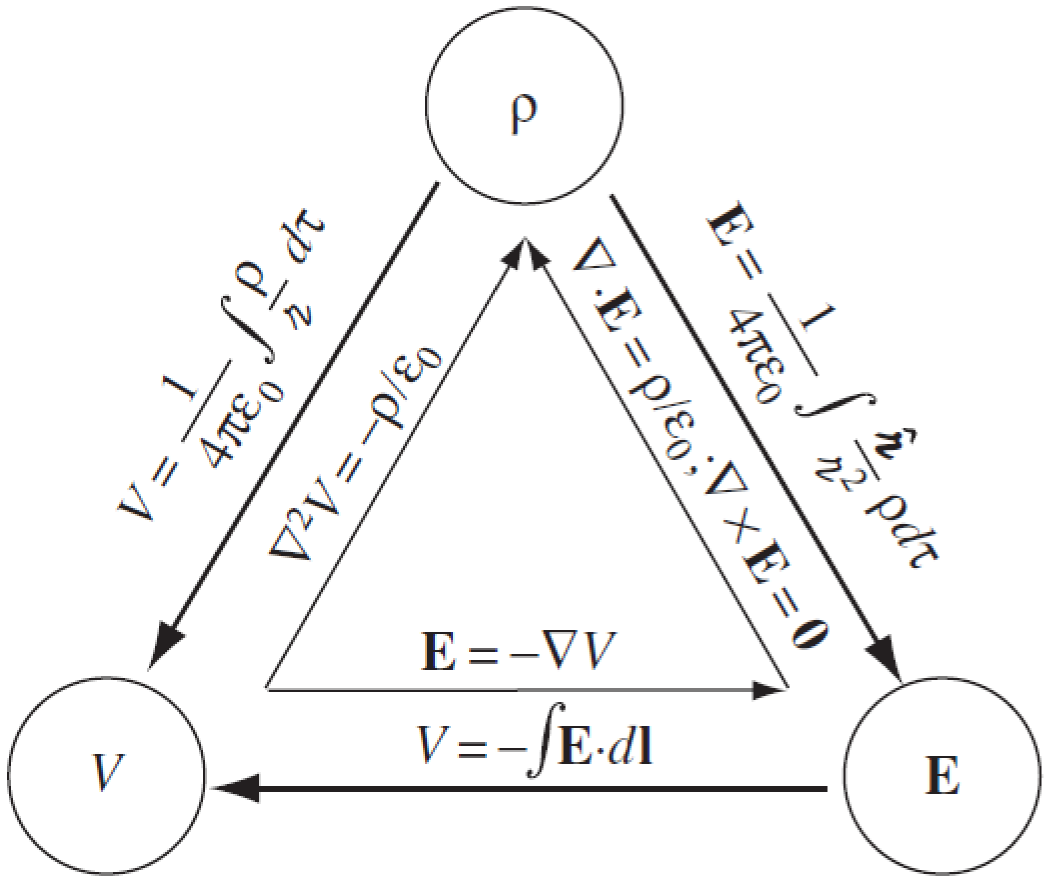
$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\nabla^2\int\left(\frac{\rho}{\mathcal{r}}\right)d\tau$$

$$= \frac{1}{4\pi\epsilon_0}\int\rho(\mathbf{r}')\left(\nabla^2\frac{1}{\mathcal{r}}\right)d\tau$$

$$= \frac{1}{4\pi\epsilon_0}\int\rho(\mathbf{r}')[-4\pi\delta^3(\mathbf{r}-\mathbf{r}')]\,d\tau$$

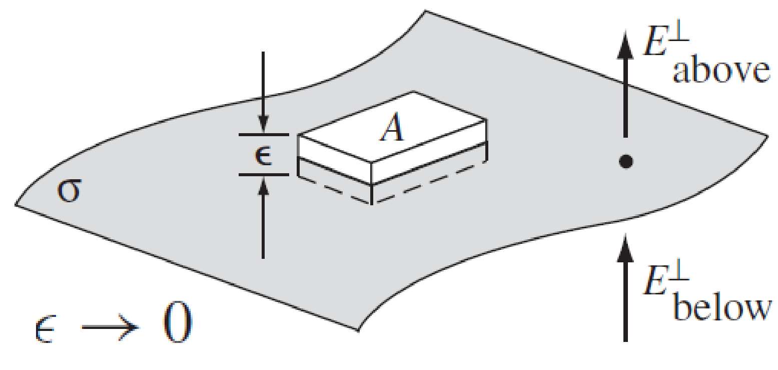
$$= -\frac{1}{\epsilon_0}\rho(\mathbf{r})$$

# Fundamental Relations of Electrostatics



# Boundary Conditions for E

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma$$

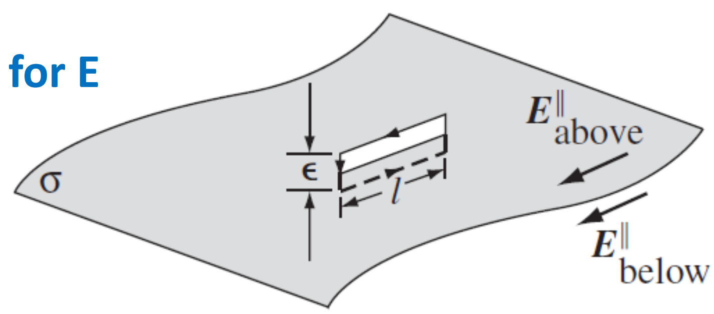



$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \Rightarrow E^{\parallel}_{\text{above}} l - E^{\parallel}_{\text{below}} l$$

$$\mathbf{E}^{\parallel}_{\text{above}} = \mathbf{E}^{\parallel}_{\text{below}}$$

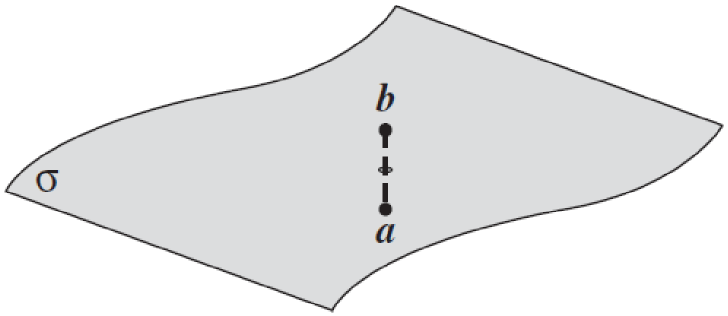
Single formula for E

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$



# Boundary Conditions for V

$$V_{\text{above}} - V_{\text{below}} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}$$




as the path length shrinks to zero

$$V_{\text{above}} = V_{\text{below}}$$

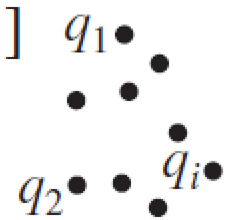
*gradient* of $V$ inherits the discontinuity in $\mathbf{E}$; since $\mathbf{E} = -\nabla V$

$$\nabla V_{\text{above}} - \nabla V_{\text{below}} = -\frac{1}{\epsilon_0}\sigma\,\hat{\mathbf{n}}$$

$$\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{1}{\epsilon_0}\sigma \quad \text{where} \quad \frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}$$

# Work and Energy

$$W = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{l} = -Q \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = Q[V(\mathbf{b}) - V(\mathbf{a})]$$





bring $Q$ in from far away and stick it at point $\mathbf{r}$

$$W = Q[V(\mathbf{r}) - V(\infty)]$$

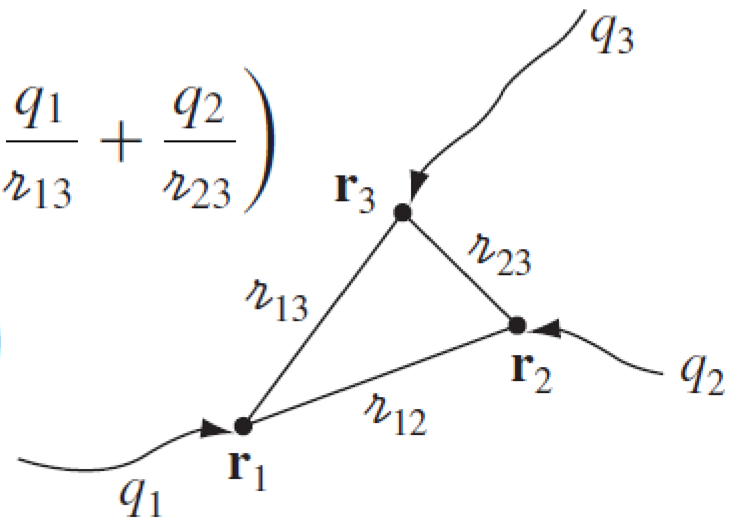
if you have set the reference point at infinity.

$$W = QV(\mathbf{r})$$

# The Energy of a Point Charge Distribution

$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left( \frac{q_1}{\mathscr{r}_{12}} \right) \qquad W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left( \frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}} \right)$$

$$W_4 = \frac{1}{4\pi\epsilon_0} q_4 \left( \frac{q_1}{\mathscr{r}_{14}} + \frac{q_2}{\mathscr{r}_{24}} + \frac{q_3}{\mathscr{r}_{34}} \right)$$




The *total* work necessary to assemble the first four charges

$$W = \frac{1}{4\pi\epsilon_0} \left( \frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_1 q_4}{\mathscr{r}_{14}} + \frac{q_2 q_3}{\mathscr{r}_{23}} + \frac{q_2 q_4}{\mathscr{r}_{24}} + \frac{q_3 q_4}{\mathscr{r}_{34}} \right)$$

# The Energy of a Point Charge Distribution

$$W = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

$j > i$ is to remind you not to count the same pair twice

*intentionally* to count each pair twice

$$W = \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\frac{q_i q_j}{\mathscr{r}_{ij}}$$

Finally, let's pull out the factor $q_i$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i\left(\sum_{j\neq i}^{n}\frac{1}{4\pi\epsilon_0}\frac{q_j}{\mathscr{r}_{ij}}\right) \Rightarrow W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i)$$

# The Energy of a Continuous Charge Distribution

$$W = \frac{1}{2}\int \rho V \, d\tau$$

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E}, \quad \text{so} \quad W = \frac{\epsilon_0}{2}\int (\nabla \cdot \mathbf{E}) V \, d\tau$$

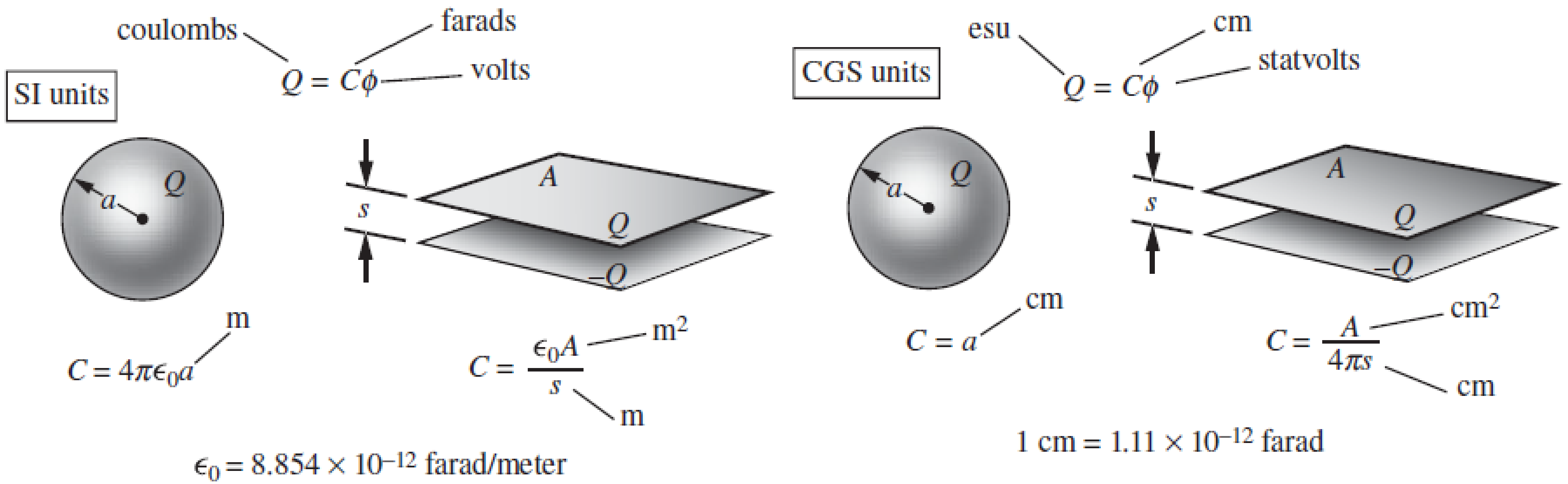
$$W = \frac{\epsilon_0}{2}\left[ -\int \mathbf{E} \cdot (\nabla V) \, d\tau + \oint V \mathbf{E} \cdot d\mathbf{a} \right]$$

$$W = \frac{\epsilon_0}{2}\left( \int_{\mathcal{V}} E^2 \, d\tau + \oint_{\mathcal{S}} V \mathbf{E} \cdot d\mathbf{a} \right) \qquad \nabla V = -\mathbf{E}$$

$$W = \frac{\epsilon_0}{2}\int E^2 \, d\tau$$

→ **(All space)**

# Capacitor




$$V = V_+ - V_- = -\int_{(-)}^{(+)} \mathbf{E} \cdot d\mathbf{l}$$

# *Capacitor*

Since **E** is proportional to $Q$, so also is $V$. The constant of proportionality is called the **capacitance** of the arrangement:

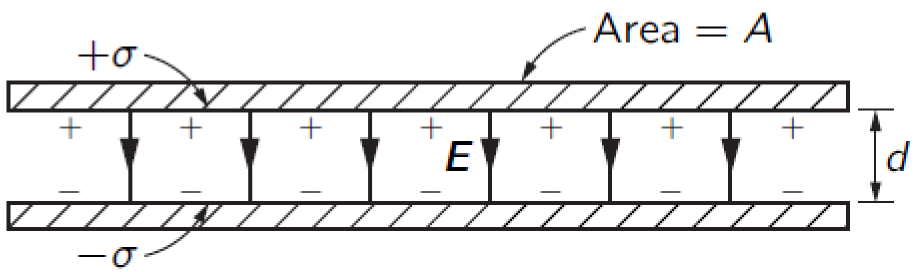
$$C \equiv \frac{Q}{V}.$$

Capacitance is a purely geometrical quantity, determined by the sizes, shapes, and separation of the two conductors. In SI units, $C$ is measured in **farads** (F); a farad is a coulomb-per-volt.

How much work does it take to charge the capacitor up to a final amount $Q$?

$$dW = \left(\frac{q}{C}\right) dq \Rightarrow \quad W = \int_0^Q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{Q^2}{C} \Rightarrow \quad W = \frac{1}{2} C V^2$$

# Capacitor



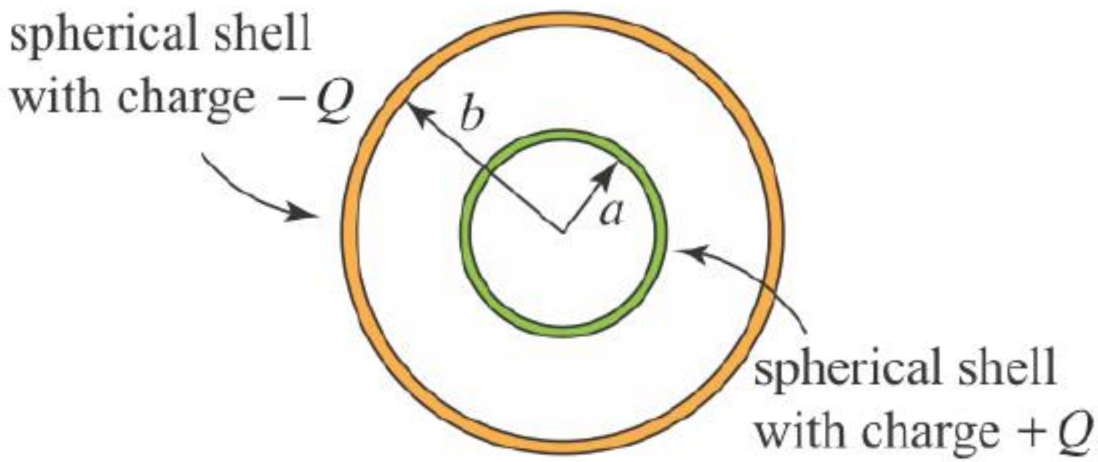

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

$$V = \frac{Q}{A\epsilon_0}d \qquad V = V_+ - V_- = -\int_{(-)}^{(+)} \mathbf{E}\cdot d\mathbf{l}$$


$$V(b) - V(a) = -\int_a^b \mathbf{E}\cdot d\mathbf{l} = -\frac{Q}{2\pi\epsilon_0 L}\int_a^b \frac{1}{s}ds = -\frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)$$

# Two spherical shells(capacitance & maximum charge) Part-a

Consider a spherical capacitor filled with air consisting of inner and outer thin conducting spherical shells with charge $+Q$ on the inner shell of radius $a = 0.10$ m and charge $-Q$ on the outer shell of radius $b = 0.2$ m. You may neglect the thickness of each shell.




**(Part a)** Find the magnitude of the electric field in each of the regions (i) $r < a$, (ii) $a < r < b$, and (iii) $r > b$.

# Two spherical shells(capacitance & maximum charge) Part-a

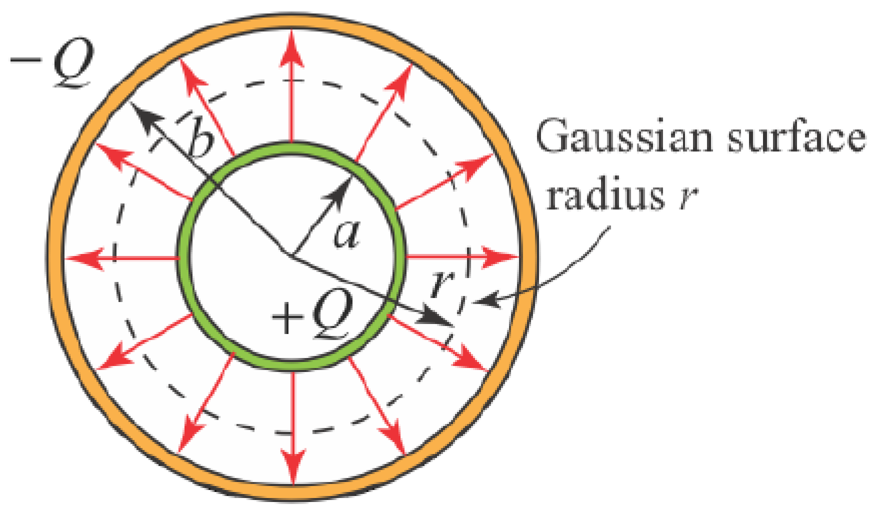
The shells have spherical symmetry so we need to use spherical Gaussian surfaces. Space is divided into three regions (I) outside , (II) in between $a < r < b$ and (III) inside $r \leq a$. In each region the electric field is purely radial (that is $\vec{\mathbf{E}} = E\hat{\mathbf{r}}$).

Region I: Outside $r \geq b$: Region III: Inside $r \leq a$:

These Gaussian surfaces contain a total charge of $0$, so the electric fields in these regions must be $0$ as well.

Region II: In between : Choose a Gaussian sphere of radius $r$. The electric flux on the surface is

$$\oiint_{\text{Gaussian sphere}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = 4\pi r^2 E = \frac{Q_{enc}}{\varepsilon_0}$$



# Two spherical shells(capacitance & maximum charge) Part-a

The enclosed charge is $Q_{enc} = +Q$, and the electric field is everywhere perpendicular to the surface. Thus Gauss's Law becomes

$$E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0} \Rightarrow E = \frac{Q}{4\pi\varepsilon_0 r^2}$$

That is, the electric field outside the inner shell is exactly the same as if all the charge of the inner shell were at the center of shell:

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}, \text{for } a < r < b \\ 0, \text{ elsewhere} \end{cases}$$

# Two spherical shells(capacitance & maximum charge) Part-b

**(Part b)** What is the electric potential difference $V(a) - V(b)$? Do you expect this potential difference to be positive or negative? Be careful to enter the correct sign for your answer.

We expect that the positively charged inner shell is at a higher potential than the negatively charged outer shell. The potential difference between the shells is

$$V(a) - V(b) = -\int_b^a \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\int_b^a \frac{Q}{4\pi\varepsilon_0 r^2} dr = \frac{Q}{4\pi\varepsilon_0 r}\bigg|_b^a = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) > 0$$

which is positive as we expect.

# Two spherical shells(capacitance & maximum charge) Part-c

**(Part c)** What is the capacitance of this object?

We calculate the capacitance using the definition

$$C = \frac{Q}{|\Delta V|} = \frac{Q}{\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)} = \frac{4\pi\varepsilon_0}{\left(\frac{1}{a} - \frac{1}{b}\right)} = \frac{4\pi\varepsilon_0 ab}{b - a}$$

Show that in the limit when the distance between the shells is very small $b - a \to \delta$, the capacitance approaches the result for a parallel plate capacitor, where $C = \varepsilon_0 A/\delta$.

If the distance between the shells is very small then the spherical capacitor begins to look very much like two parallel plates separated by a distance $\delta = b - a$ with area

$$A \approx 4\pi\left(\frac{a+b}{2}\right)^2 \approx 4\pi\left(\frac{a+a}{2}\right)^2 = 4\pi a^2 \approx 4\pi ab$$

So, in this limit, the spherical formula is the same at the capacitance of a parallel plate capacitor, $C = \varepsilon_0 A/\delta$.

$$C = lim_{b\to a}\frac{4\pi\varepsilon_0 ab}{b - a} \simeq \frac{\varepsilon_0\left(4\pi a^2\right)}{\delta} = \frac{\varepsilon_0 A}{\delta}$$

# Two spherical shells(capacitance & maximum charge) Part-d

**(Part d i)** What is the capacitance of this object in Farads?

**(Part d ii)** Suppose the maximum possible electric field at the outer surface of the inner shell before the air starts to ionize is $E(a) = 3.0 \times 10^6 V \cdot m^{-1}$. What is the maximum possible charge on the inner shell in Coulombs?

$$C = \frac{4\pi\varepsilon_0 ab}{b-a} = \frac{(0.1m)(0.2m)}{(9 \times 10^9 N \cdot m^2 \cdot C^{-2})(0.1m)} = 2.2 \times 10^{-11} F. \qquad 1/4\pi\varepsilon_0 = 9 \times 10^9 N \cdot m^2 \cdot C^{-2}.$$

The electric field at the surface of the inner shell is given by

$$E(a) = \frac{Q}{4\pi\varepsilon_0 a^2}$$

Therefore the maximum charge on the inner shell is

$$Q_{max} = 4\pi\varepsilon_0 E_{max}(a)a^2 = \frac{(3.0 \times 10^6 V \cdot m^{-1})(0.1m)^2}{(9 \times 10^9 N \cdot m^2 \cdot C^{-2})} = 3.3 \times 10^{-6} C$$

# Two spherical shells(capacitance & maximum charge) Part-e

**(Part e i)** What is the magnitude of the maximum potential difference, $|V(a) - V(b)|$, between the shells?

**(Part e ii)** What is the maximum amount of energy stored in this capacitor?

We can find the potential difference two different ways. Using the definition of capacitance we have that

$$V(a) - V(b) = \frac{Q}{C} = \frac{4\pi\varepsilon_0 E(a)a^2(b-a)}{4\pi\varepsilon_0 ab} = \frac{E(a)a(b-a)}{b}$$

$$V(a) - V(b) = \frac{(3.0 \times 10^6 V \cdot m^{-1})(0.1m)(0.1m)}{(0.2m)} = 1.5 \times 10^5 V$$

The second method relies on our calculation of the potential difference in part b):

$$\Delta V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$U_{max} = \frac{Q_{max}^2}{2C} = \frac{(3.3 \times 10^{-6} C)^2}{(2)(2.2 \times 10^{-11} F)} = 2.5 \times 10^{-1} J$$

Recall that $E(a) = \frac{Q}{4\pi\varepsilon_0 a^2}$ or $\frac{Q}{4\pi\varepsilon_0} = E(a)a^2$. Substitute this into our expression for potential difference yielding

$$V(a) - V(b) = E(a)a^2\left(\frac{1}{a} - \frac{1}{b}\right) = E(a)a^2\frac{b-a}{ab} = E(a)a\frac{b-a}{b}$$

# Two spherical shells(capacitance & maximum charge) Part-f

**(Part f)** Now, with the charge still at its maximum possible value, consider the case that the dimension of the outer shell is doubled from $b$ to $2b$. What is the change in the stored energy? Assume that the charge on the shells is not changed.

In terms of charge and capacitance, the change in the stored energy is given by

$$\Delta U = U_f - U_i = \frac{Q^2}{2C_f} - \frac{Q^2}{2C_i}$$

$$= \frac{1}{2}\left(\frac{Q^2}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{2b}\right)\right) - \frac{1}{2}\left(\frac{Q^2}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)\right)$$

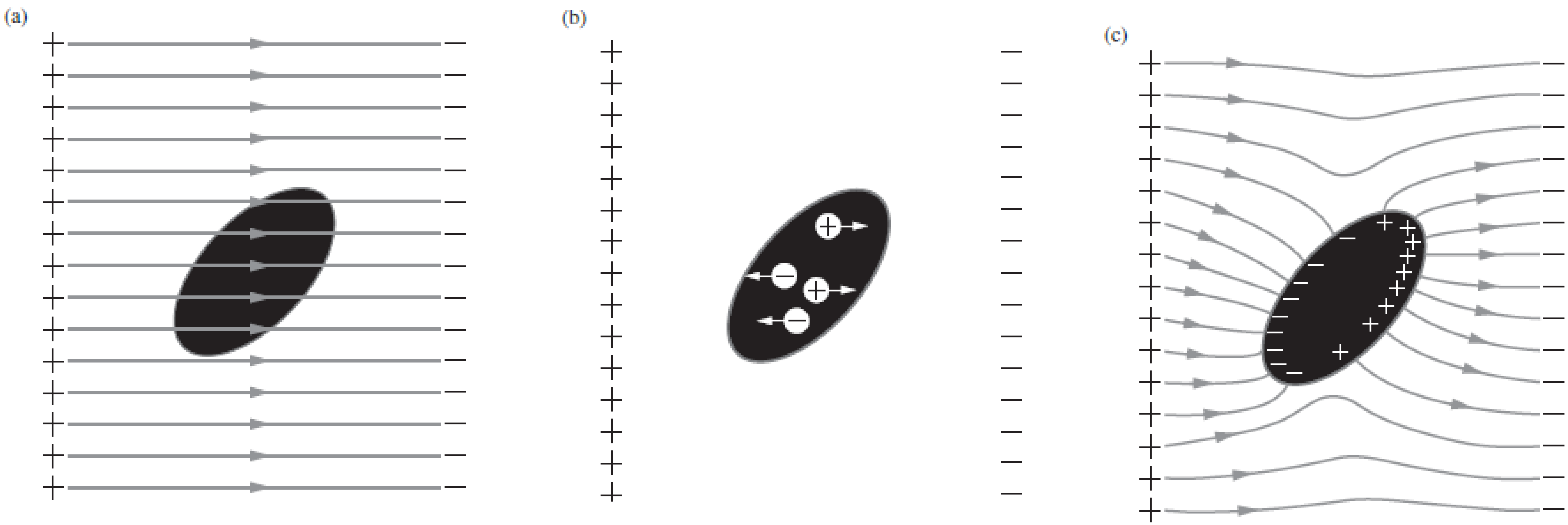
$$= \frac{Q^2}{16\pi\varepsilon_0 b}$$

The maximum change in stored energy is when $Q_{max} = 3.3 \times 10^{-6} C$. Therefore

$$\Delta U_{max} = \frac{Q_{max}^2}{16\pi\varepsilon_0 b} = \frac{1}{4}\frac{(3.3 \times 10^{-6} C)^2 (9 \times 10^9 N \cdot m^2 \cdot C^{-2})}{(0.2 m)} = 1.23 \times 10^{-1} J$$

Since the oppositely charged spheres attract each other, it make sense that it takes positive energy to move the outer shell farther away.

# Conductors in the electrostatic field




The object in (a) is a neutral nonconductor. The charges in it, both positive and negative, are immobile. In (b) the charges have been released and begin to move. They will move until the final condition, shown in (c), is attained.

# Conductors in the electrostatic field

$\mathbf{E} = 0$ inside the material of a conductor;

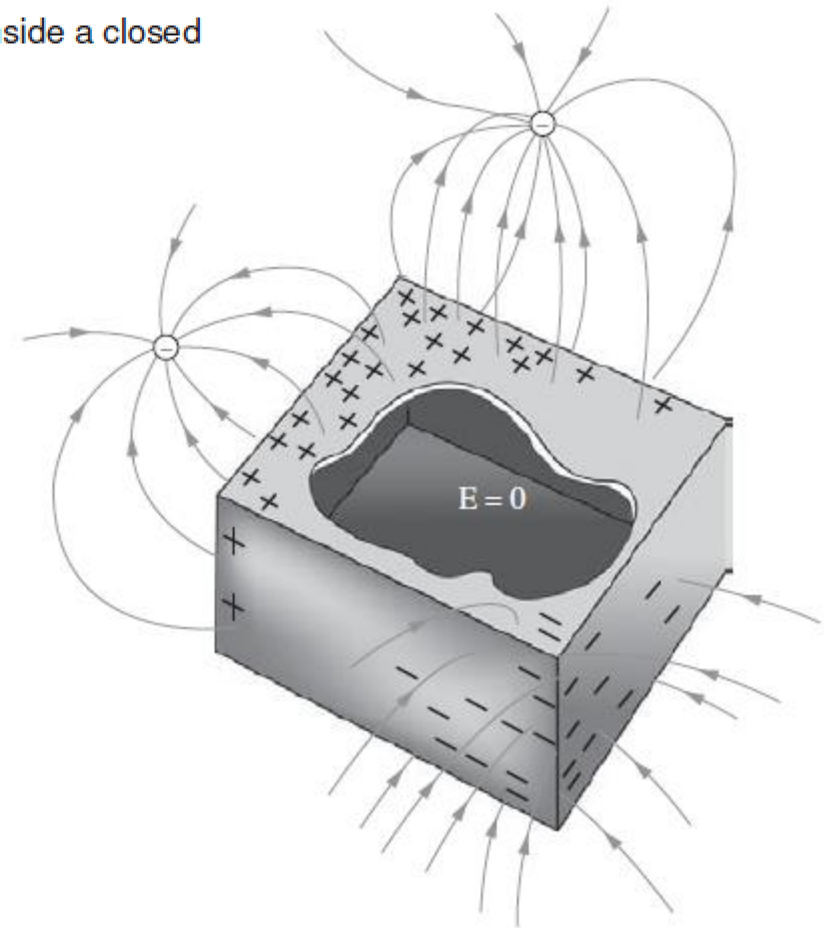
$\rho = 0$ inside the material of a conductor;

At any point just outside the conductor, $\mathbf{E}$ is perpendicular to the surface, and $E = \sigma/\epsilon_0$, where $\sigma$ is the local density of surface charge;

Because the surface of a conductor is necessarily a surface of constant potential, the electric field, which is $-\text{grad}\,\phi$, must be *perpendicular* to the surface at every point on the surface.

# Conductors in the electrostatic field

The field is zero everywhere inside a closed conducting box.

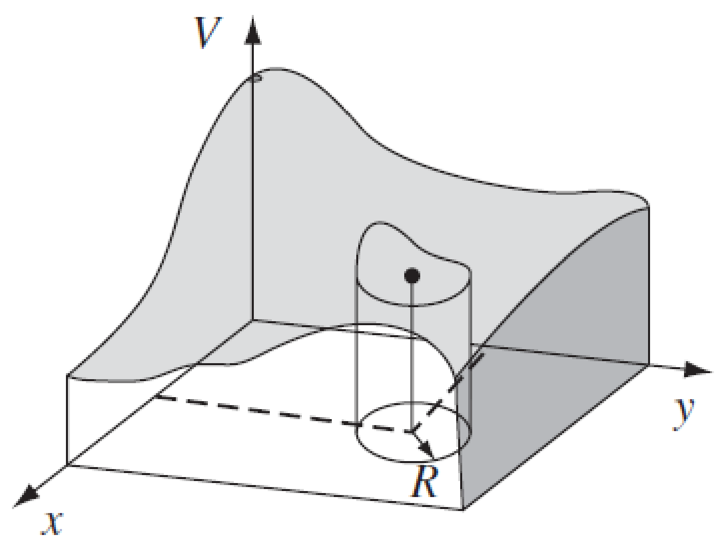


# Laplace's Equation (2-D)

1. The value of $V$ at a point $(x, y)$ is the average of those *around* the point. More precisely, if you draw a circle of any radius $R$ about the point $(x, y)$, the average value of $V$ on the circle is equal to the value at the center:

$$V(x, y) = \frac{1}{2\pi R} \oint_{\text{circle}} V \, dl.$$

2. $V$ has no local maxima or minima; all extrema occur at the boundaries.



# Laplace's Equation (3-D)

1. The value of $V$ at point $\mathbf{r}$ is the average value of $V$ over a spherical surface of radius $R$ centered at $\mathbf{r}$:

$$V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V \, da.$$

2. As a consequence, $V$ can have no local maxima or minima; the extreme values of $V$ must occur at the boundaries.

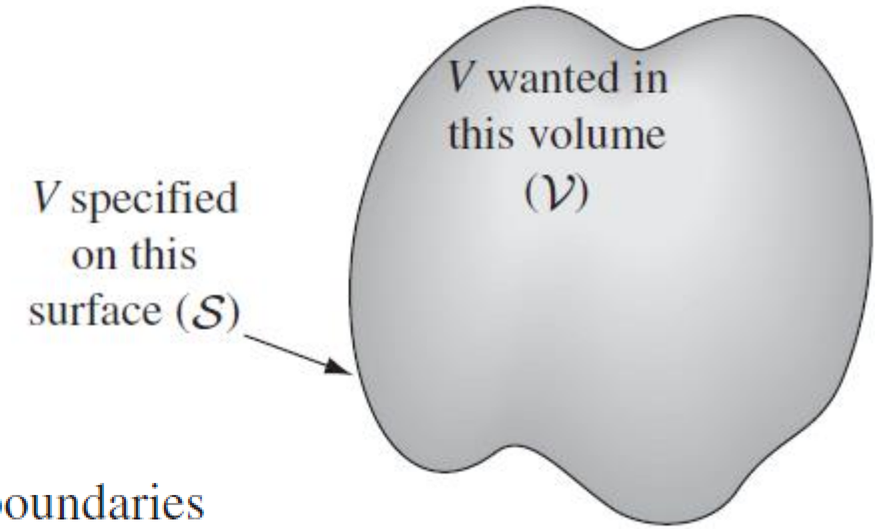
# Boundary conditions and Uniqueness Theorems

**First uniqueness theorem:** The solution to Laplace's equation in some volume $\mathcal{V}$ is uniquely determined if $V$ is specified on the boundary surface $\mathcal{S}$.

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0$$

were *two* solutions to Laplace's equation

$$V_3 \equiv V_1 - V_2.$$




$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0,$$

obeys Laplace's equation

and it takes the value *zero* on all boundaries

Laplace's equation allows no local maxima or minima-

So the maximum and minimum of $V_3$ are both zero.

Therefore $V_3$ must be zero everywhere, and hence $V_1 = V_2$.

# Conductors (Second Uniqueness Theorem)

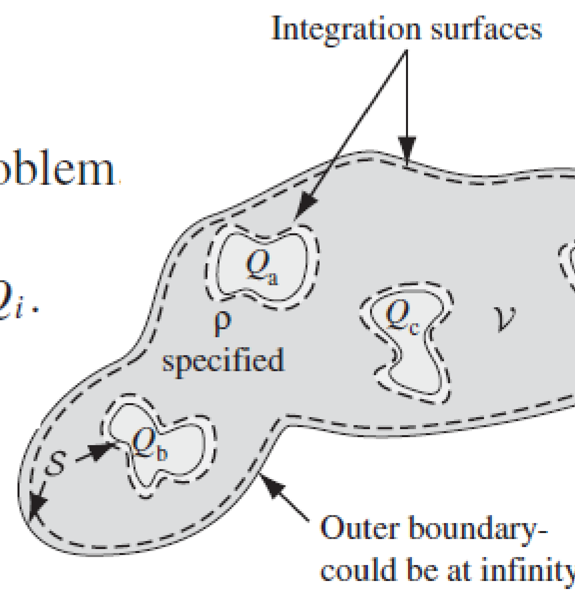
**Second uniqueness theorem:** In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density $\rho$, the electric field is uniquely determined if the *total charge* on each conductor is given

$$\nabla \cdot \mathbf{E}_1 = \frac{1}{\epsilon_0}\rho, \qquad \nabla \cdot \mathbf{E}_2 = \frac{1}{\epsilon_0}\rho.$$

*two* fields satisfying the conditions of the problem

$$\oint_{\substack{i\text{th conducting}\\ \text{surface}}} \mathbf{E}_1 \cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q_i, \qquad \oint_{\substack{i\text{th conducting}\\ \text{surface}}} \mathbf{E}_2 \cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q_i.$$

$$\oint_{\substack{\text{outer}\\ \text{boundary}}} \mathbf{E}_1 \cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q_{\text{tot}}, \qquad \oint_{\substack{\text{outer}\\ \text{boundary}}} \mathbf{E}_2 \cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q_{\text{tot}}.$$



# Conductors (Second Uniqueness Theorem)

$$\mathbf{E}_3 \equiv \mathbf{E}_1 - \mathbf{E}_2, \qquad \text{which obeys} \qquad \nabla \cdot \mathbf{E}_3 = 0$$

in the region between the conductors, and $\oint \mathbf{E}_3 \cdot d\mathbf{a} = 0$

over each boundary surface.

each conductor is an equipotential, and hence $V_3$ is a *constant*

$$\nabla \cdot (V_3 \mathbf{E}_3) = V_3(\nabla \cdot \mathbf{E}_3) + \mathbf{E}_3 \cdot (\nabla V_3) = -(E_3)^2.$$

$$\int_{\mathcal{V}} \nabla \cdot (V_3 \mathbf{E}_3)\, d\tau = \oint_{\mathcal{S}} V_3 \mathbf{E}_3 \cdot d\mathbf{a} = -\int_{\mathcal{V}} (E_3)^2\, d\tau.$$

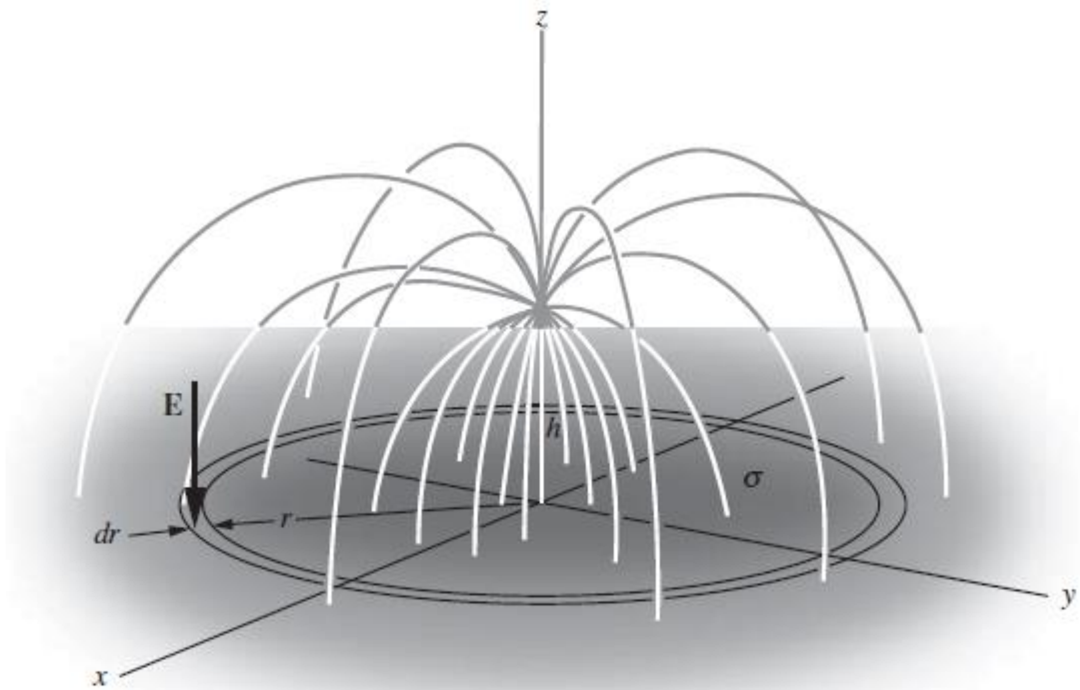
The surface integral covers all boundaries of the region in question—the conductors and outer boundary. Now $V_3$ is a constant over each surface (if the outer boundary is infinity, $V_3 = 0$ there), so it comes outside each integral, and what remains is zero,

# Conductors (Second Uniqueness Theorem)

$$\int_{\mathcal{V}} (E_3)^2 \, d\tau = 0.$$

But this integrand is never negative; the only way the integral can vanish is if $E_3 = 0$ everywhere. Consequently, $\mathbf{E}_1 = \mathbf{E}_2$

# The method of Images



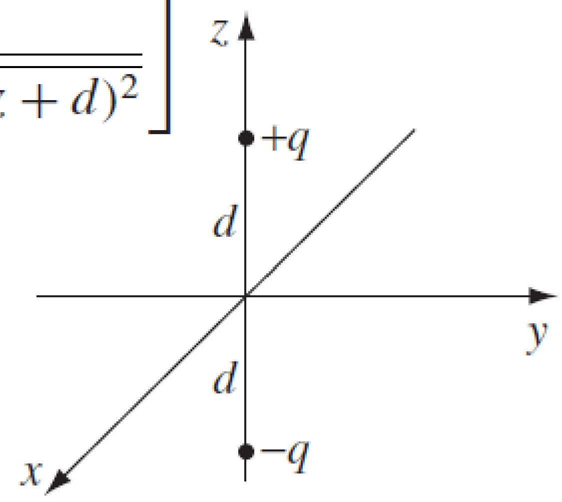

Suppose a point charge $q$ is held a distance $d$ above an infinite grounded conducting plane *Question:* What is the potential in the region above the plane?

# Image Charges

From a mathematical point of view, our problem is to solve Poisson's equation in the region $z > 0$, with a single point charge $q$ at $(0, 0, d)$, subject to the boundary conditions:

1. $V = 0$ when $z = 0$ (since the conducting plane is grounded), and
2. $V \to 0$ far from the charge $\left(\text{that is, for } x^2 + y^2 + z^2 \gg d^2\right)$.

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z - d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z + d)^2}}\right]$$

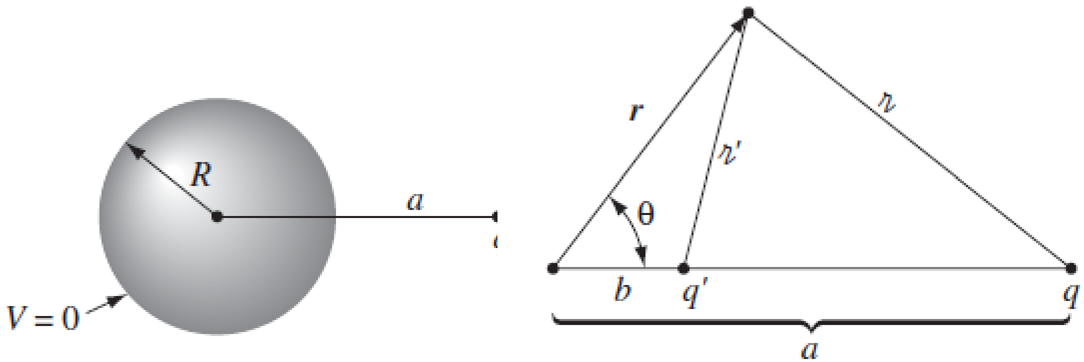



1. $V = 0$ when $z = 0$,
2. $V \to 0$ for $x^2 + y^2 + z^2 \gg d^2$

# Image Charges

A point charge $q$ is situated a distance $a$ from the center of a grounded conducting sphere of radius $R$

Find the potential outside the sphere.



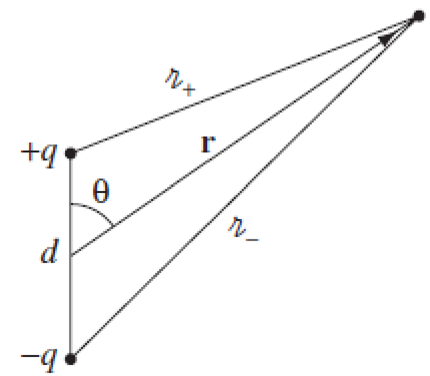

$$b = \frac{R^2}{a}$$

$$q' = -\frac{R}{a} q$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right)$$

this potential vanishes at all points on the sphere, and therefore fits the boundary conditions for our original problem, in the exterior region.

# Dipoles




$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$$

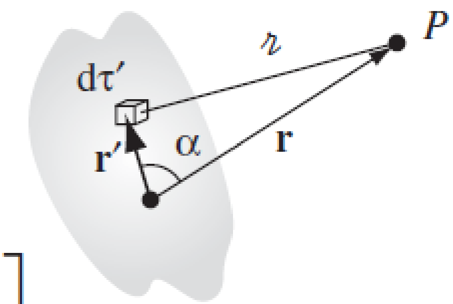
$$\mathscr{r}_\pm^2 = r^2 + (d/2)^2 \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

$$\frac{1}{\mathscr{r}_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2} \cong \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

$$\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} \cong \frac{d}{r^2}\cos\theta,$$

$$V(\mathbf{r}) \cong \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}.$$

# Multipole Expansion (Approximate potential at Large distance)




$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau'.$$

Using the law of cosines,

$$\mathscr{r}^2 = r^2 + (r')^2 - 2rr'\cos\alpha = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

where $\alpha$ is the angle between $\mathbf{r}$ and $\mathbf{r}'$. Thus $\mathscr{r} = r\sqrt{1+\epsilon}$, $\quad \epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$.

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \ldots\right)$$

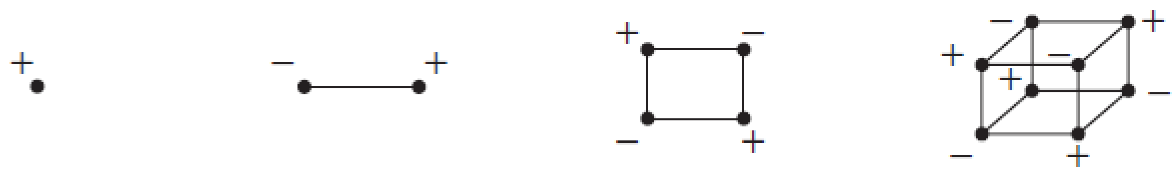
$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)(\cos\alpha) + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right) + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\alpha - 3\cos\alpha}{2}\right) + \ldots\right]$$

# Multipole Expansion

$$\frac{1}{\mathscr{r}} = \frac{1}{r} \sum_{n=0}^{\infty} \left( \frac{r'}{r} \right)^n P_n(\cos\alpha).$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{(n+1)}} \int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\, d\tau'$$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{1}{r} \int \rho(\mathbf{r}')\, d\tau' + \frac{1}{r^2} \int r' \cos\alpha\, \rho(\mathbf{r}')\, d\tau' \right.$$
$$\left. + \frac{1}{r^3} \int (r')^2 \left( \frac{3}{2}\cos^2\alpha - \frac{1}{2} \right) \rho(\mathbf{r}')\, d\tau' + \dots \right].$$




Monopole
$(V \sim 1/r)$

Dipole
$(V \sim 1/r^2)$

Quadrupole
$(V \sim 1/r^3)$

Octopole
$(V \sim 1/r^4)$

# Legendre Polynomials

$$P_l(x) \equiv \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l.$$

Rodrigues formula

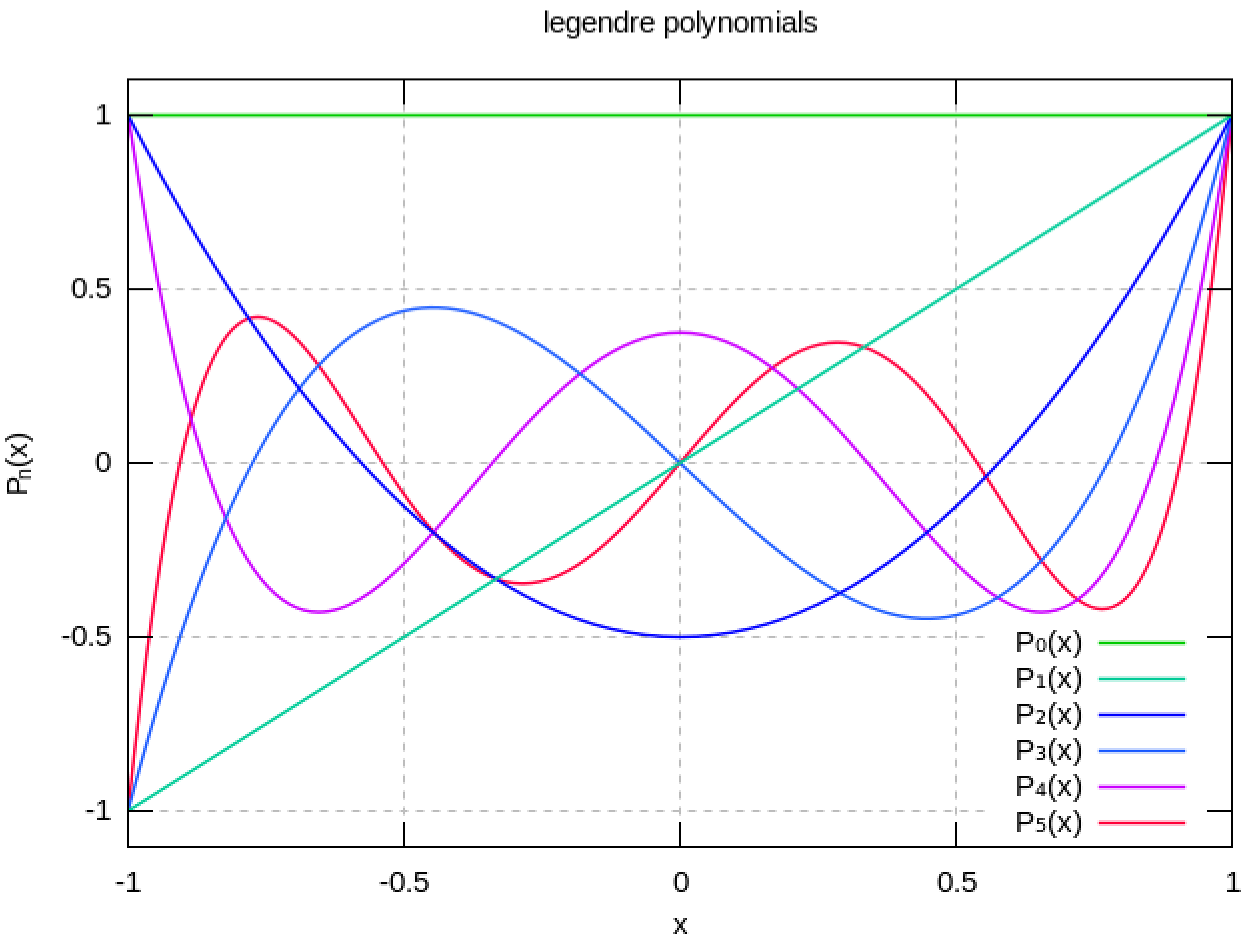
$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= (3x^2-1)/2 \\
P_3(x) &= (5x^3-3x)/2 \\
P_4(x) &= (35x^4-30x^2+3)/8 \\
P_5(x) &= (63x^5-70x^3+15x)/8
\end{aligned}$$

# Legendre Polynomials (Orthogonality)

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\, dx = \int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta$$

$$= \begin{cases} 0, & \text{if } l' \neq l, \\ \dfrac{2}{2l+1}, & \text{if } l' = l. \end{cases}$$

# Legendre Polynomials (Plot)



# The monopole and dipole terms

$$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\alpha\,\rho(\mathbf{r}')\,d\tau'$$ Since $\alpha$ is the angle between $\mathbf{r}'$ and $\mathbf{r}$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}\cdot\int \mathbf{r}'\rho(\mathbf{r}')\,d\tau'$$ $$r'\cos\alpha = \hat{\mathbf{r}}\cdot\mathbf{r}'$$

$$\mathbf{p} \equiv \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau'$$ **dipole moment** of the distribution:

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

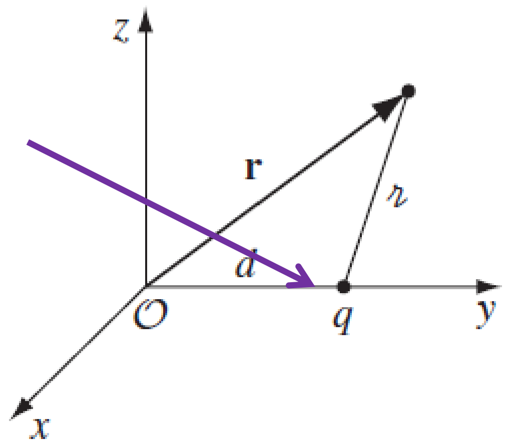
Thus, the dipole moment of a collection of *point* charges is $$\mathbf{p} = \sum_{i=1}^{n} q_i\mathbf{r}'_i.$$

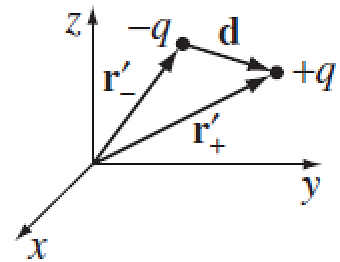
# Dipole Electric field

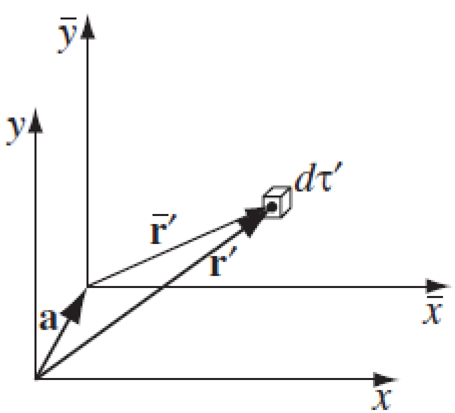
A *physical* dipole becomes a *pure* dipole, then, in the rather artificial limit

$d \to 0$, $q \to \infty$, with the product $qd = p$ held fixed.

Monopole moment Q does not change with the shift of charge q from the origin



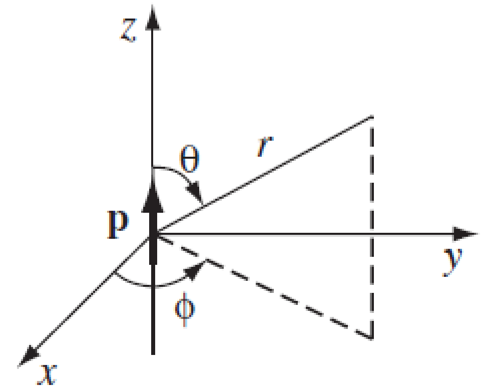

$$\bar{\mathbf{p}} = \int \bar{\mathbf{r}}'\rho(\mathbf{r}')\,d\tau' = \int (\mathbf{r}' - \mathbf{a})\rho(\mathbf{r}')\,d\tau'$$

$$= \int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' - \mathbf{a}\int \rho(\mathbf{r}')\,d\tau' = \mathbf{p} - Q\mathbf{a}.$$

Dipole moment does change with the shift of origin but an exception happens when the total charge is zero.

# Dipole Electric field

$$V_{dip}(r,\theta) = \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}.$$




$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

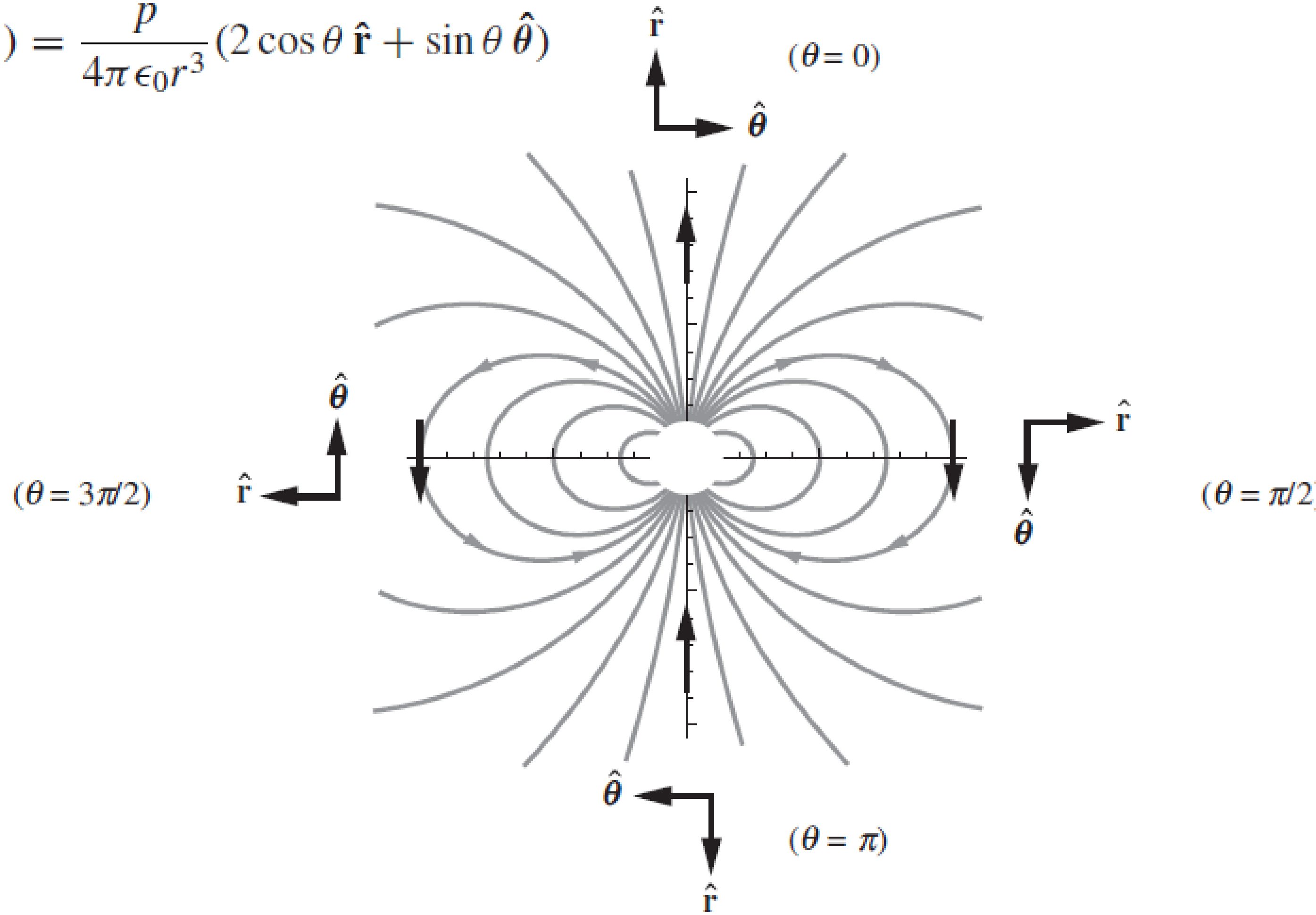
$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = 0$$

$$\mathbf{E}_{dip}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

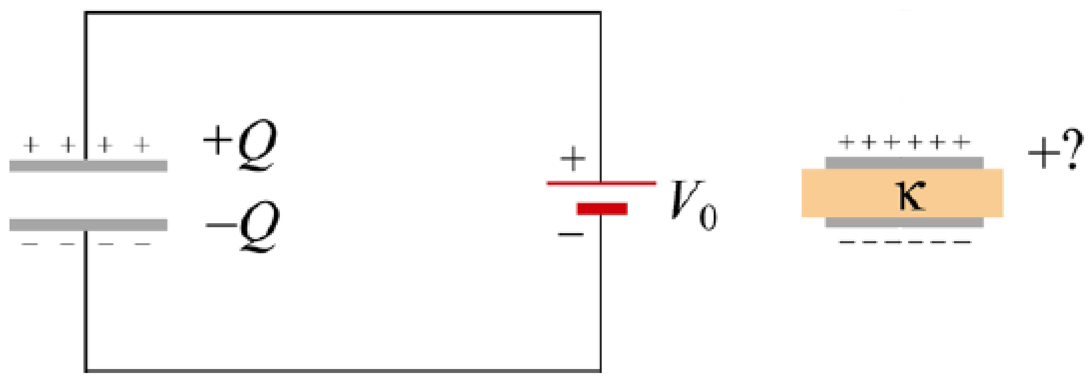
# Dipole Electric field

$$\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$



# Insert Dielectric with Battery disconnected- Charge

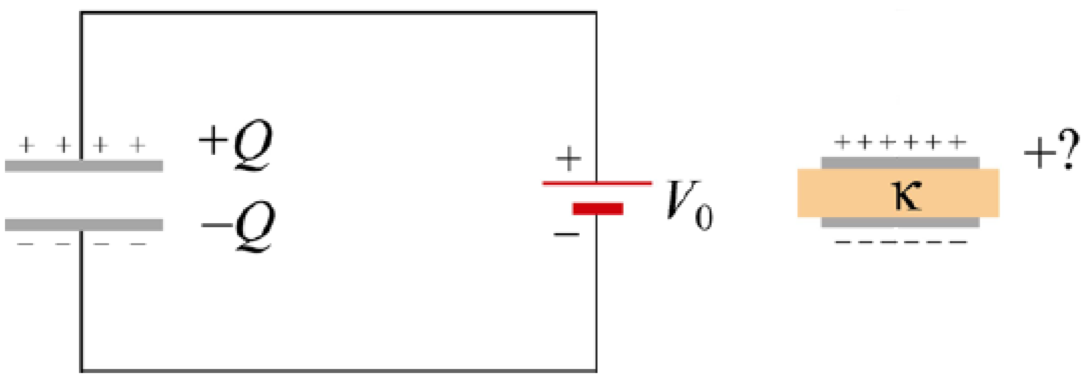
A parallel plate capacitor is charged with a battery to a total charge $Q$ on the positive plate and the battery is removed. A slab of material with dielectric constant $\kappa$ is inserted between the plates. The charge on the positive plate of the capacitor




- increases.
- decreases.
- stays the same.

# Insert Dielectric with Battery disconnected - Charge

A parallel plate capacitor is charged with a battery to a total charge $Q$ on the positive plate and the battery is removed. A slab of material with dielectric constant $\kappa$ is inserted between the plates. The charge on the positive plate of the capacitor

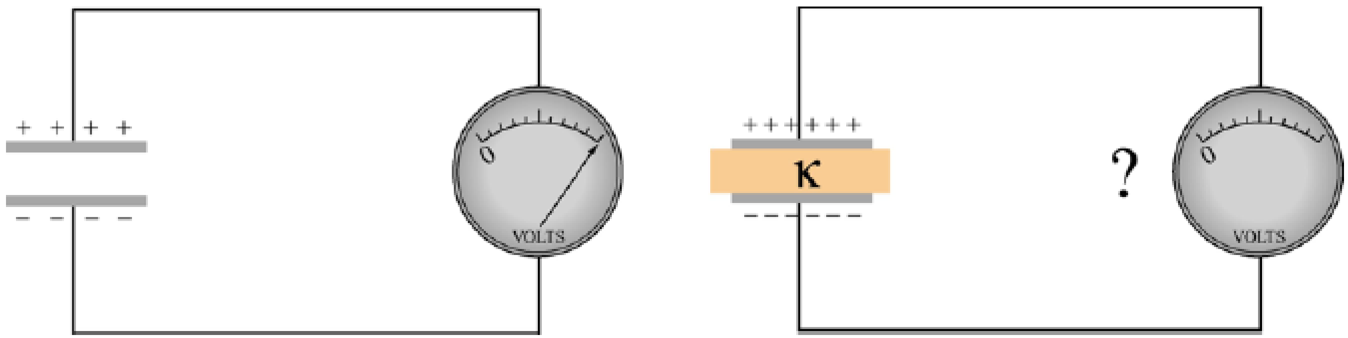



- increases.
- decreases.
- stays the same. ✔

If there is no wire or other conducting connection between the top and bottom plates of the capacitor, the charges on the plates cannot move and therefore they don't change when the dielectric is inserted.

# *Insert Dielectric with Battery disconnected- Potential*

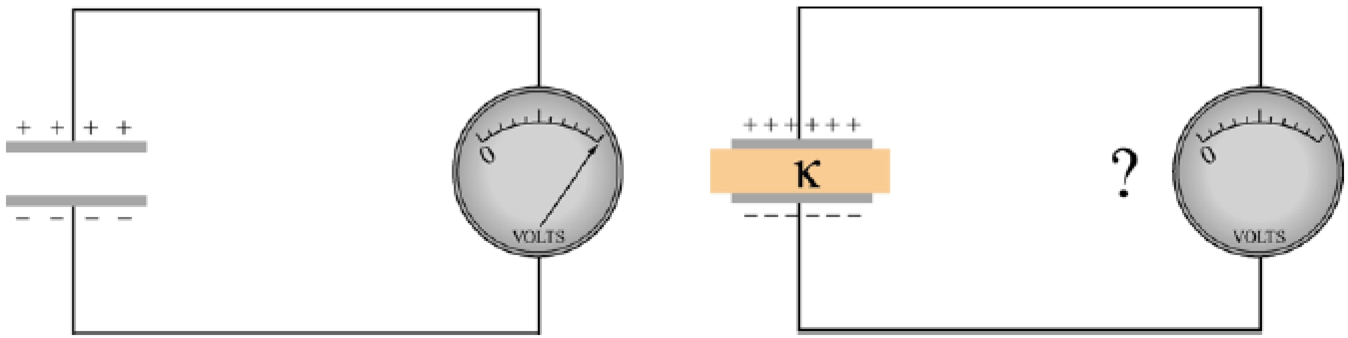
A parallel plate capacitor is charged with a battery to a total charge $Q$ on the positive plate and the battery is removed. A slab of material with dielectric constant $\kappa$ is inserted between the plates. The grey circle represents a meter which measures the voltage between the top and bottom plates. It is not a battery that supplies voltage or a wire connecting the two plates. The potential difference $V$ across the plates of the capacitor




- increases.
- decreases.
- stays the same.

# *Insert Dielectric with Battery disconnected- Potential*

A parallel plate capacitor is charged with a battery to a total charge $Q$ on the positive plate and the battery is removed. A slab of material with dielectric constant $\kappa$ is inserted between the plates. The grey circle represents a meter which measures the voltage between the top and bottom plates. It is not a battery that supplies voltage or a wire connecting the two plates. The potential difference $V$ across the plates of the capacitor

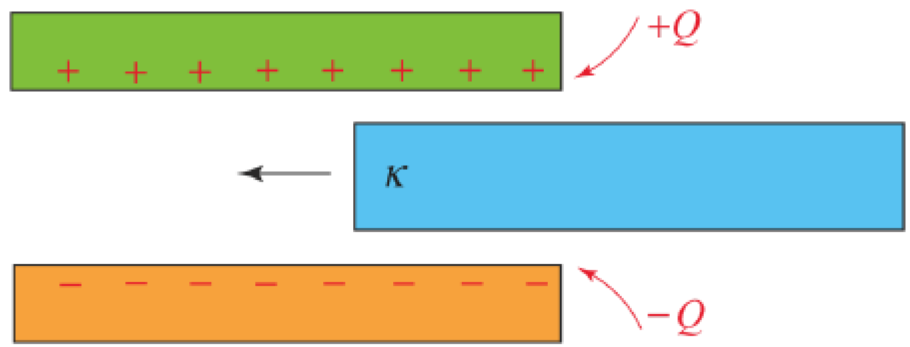



- increases.
- decreases. ✔
- stays the same.

If there is no wire or other conducting connection between the top and bottom plates of the capacitor, the charges on the plates cannot move and therefore they don't change when the dielectric is inserted. However, the dielectric does reduce the electric field created between the plates by those charges by a factor of $1/\kappa$. If $\vec{E}$ goes down, the voltage between the plates must also go down.

# Dielectric Capacitor Energy - 1

A parallel plate capacitor is charged to a total charge $Q$ and the battery is disconnected. A slab of material with dielectric constant $\kappa$ is inserted between the plates. The energy stored in the capacitor




- increases.
- decreases.
- stays the same.

# Dielectric Capacitor Energy - 1

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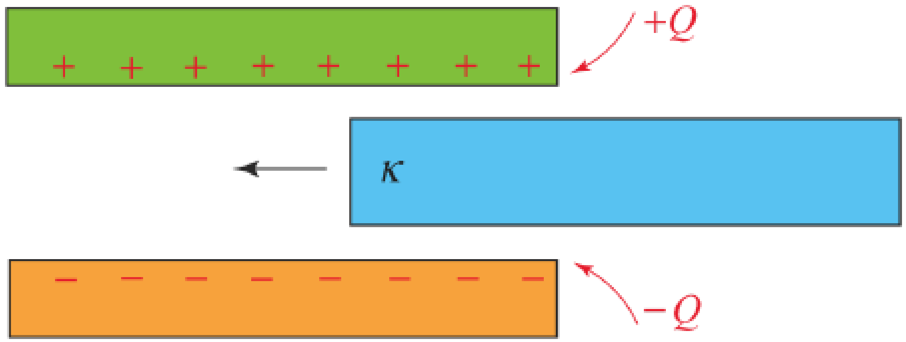



- increases.
- decreases. ✔
- stays the same.

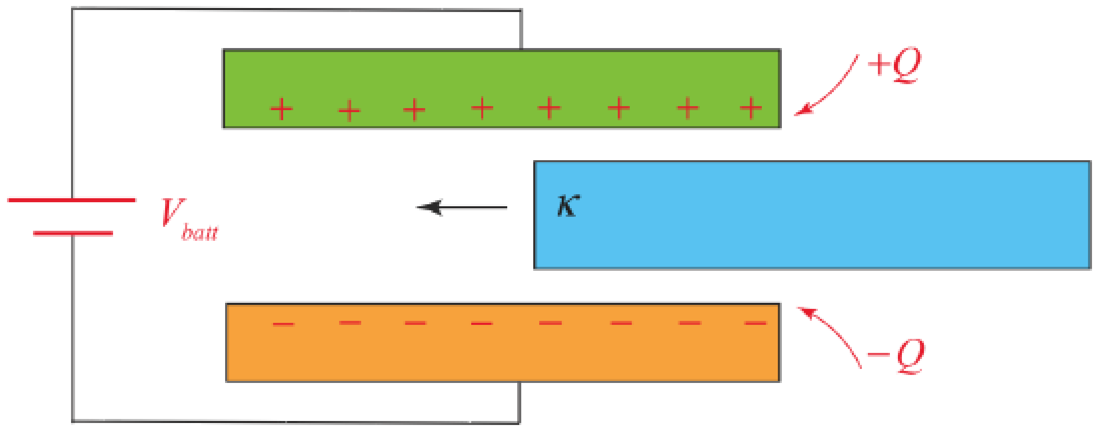
The energy stored decreases. With no connection between the plates, the charge does not change. However, the dielectric reduces the electric field and hence reduces the amount of energy stored in the field.

Another way to think about this is that the dielectric increases the capacitance while the charge remains the same. Using the equation $U = Q^2/2C$, we can see that the potential energy drops.

Why does the energy go down? The electric field induces positive and negative charges on the bottom and top plates of the dielectric, respectively. Those induced charges are attracted by the opposite charges on the capacitor plates and so the dielectric is pulled into the capacitor. That force could be used to do work or could create kinetic energy of the dielectric as it moves in. That energy must come from somewhere, so the energy stored in the capacitor must go down.

# Dielectric Capacitor Energy 2

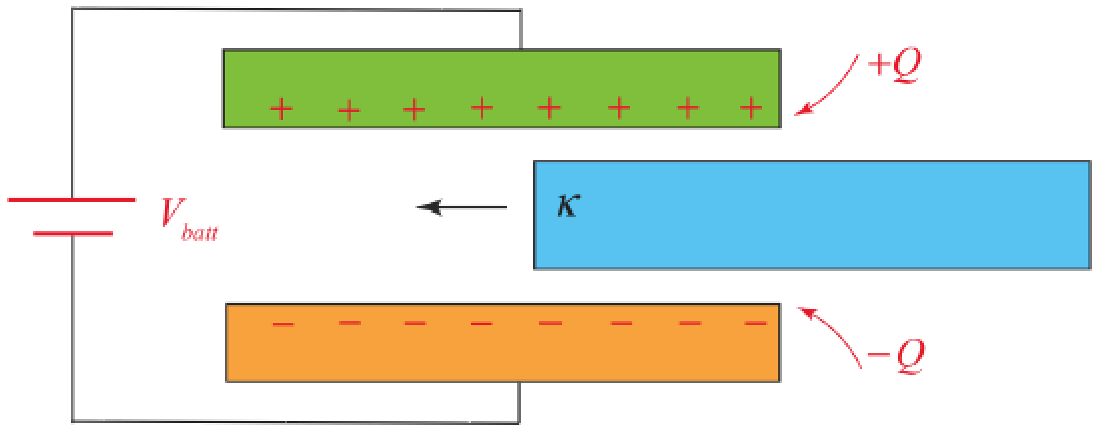
A parallel plate capacitor is charged to a total charge $Q$. While the battery is still connected a slab of material with dielectric constant $\kappa$ is inserted between the plates. The energy stored in the capacitor




- increases.
- decreases.
- stays the same.

# Dielectric Capacitor Energy 2

A parallel plate capacitor is charged to a total charge $Q$. While the battery is still connected a slab of material with dielectric constant $\kappa$ is inserted between the plates. The energy stored in the capacitor

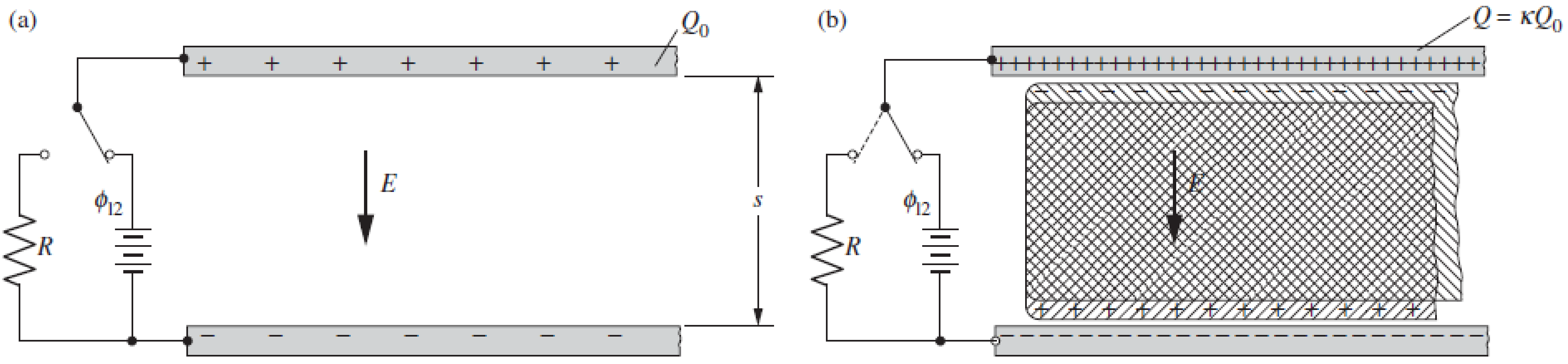



- increases. ✔
- decreases.
- stays the same.

The energy stored increases. After the dielectric is inserted, the potential difference between the plates remains the same because the plates are connected to the battery. The capacitance, however, increases due to the dielectric material. Using the equation $U = CV_{batt}^2/2$, we can see that the energy increases. Compared to the situation before the dielectric is inserted, the voltage is the same and the capacitance increases from $C$ to $\kappa C$

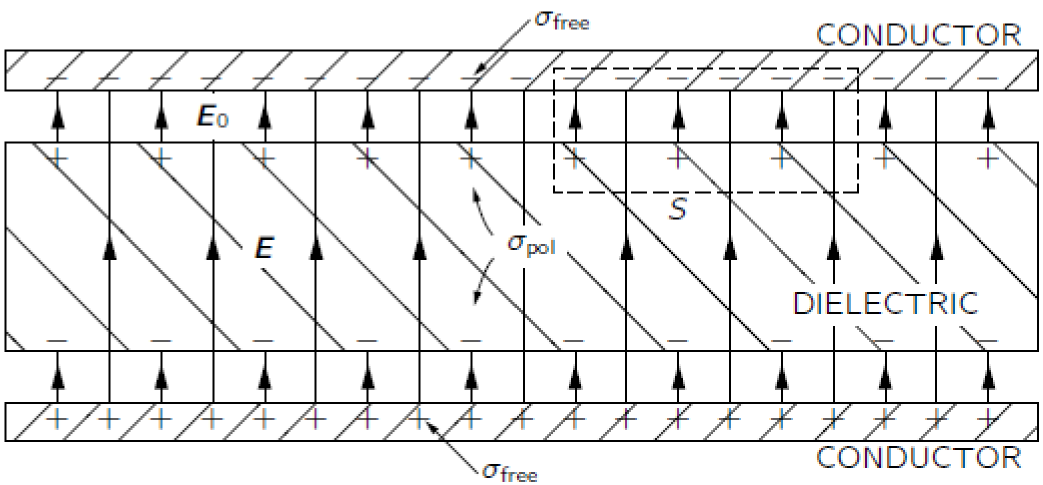
Where does that increase in energy come from? If the voltage is fixed and the capacitance goes up, then the charge on the capacitor $Q = CV_{batt}$ must also go up. The battery needs to supply energy to move this additional charge across the potential difference between the plates.

# Dielectrics



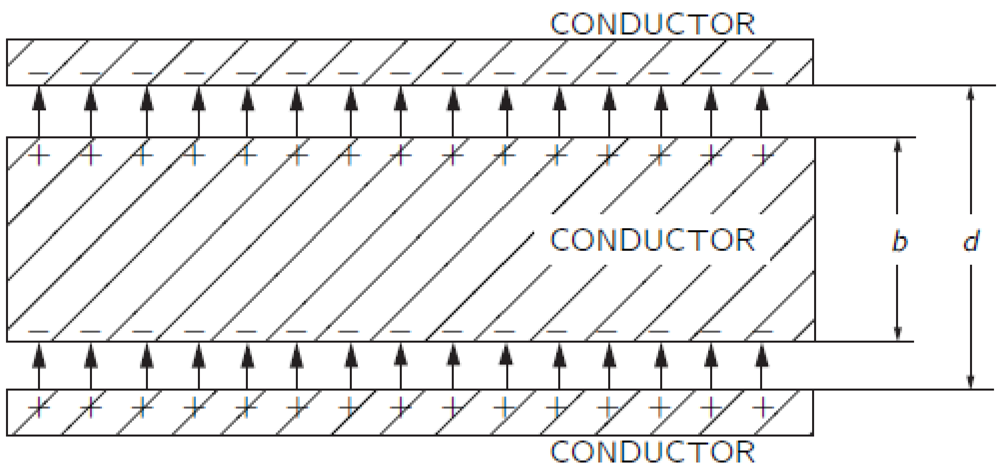

$$Q = \kappa Q_0 \quad \Longleftrightarrow \quad C = \kappa C_0$$

# Capacitor (dielectric)




Now the experimental fact is that if we put a piece of insulating material like lucite or glass between the plates, we find that the capacitance is larger. That means, of course, that the voltage is lower for the same charge. But the voltage difference is the integral of the electric field across the capacitor; so we must conclude that inside the capacitor, the electric field is reduced even though the charges on the plates remain unchanged.

# Capacitor



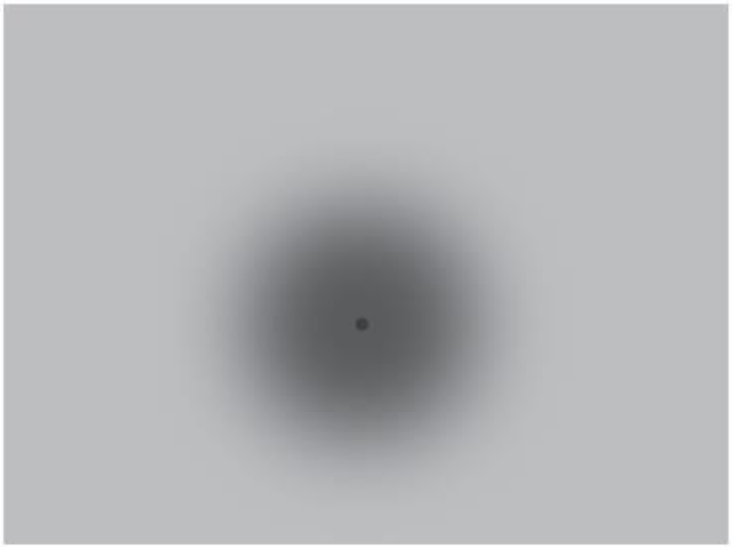

If we put a conducting plate in the gap of a parallel-plate condenser, the induced charges reduce the field in the conductor to zero.

# Dielectric Constant

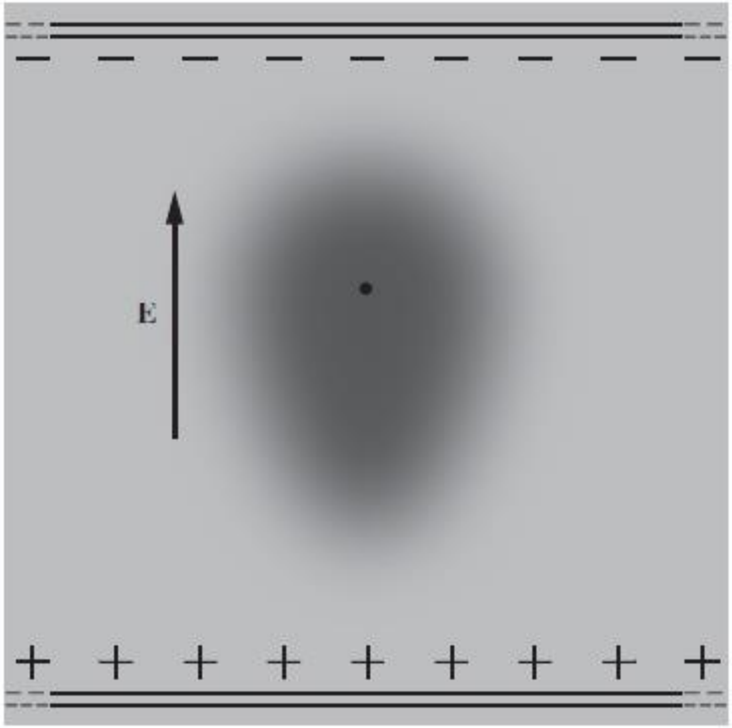
Dielectric constants of various substances

| Substance | Conditions | Dielectric constant ($\kappa$) |
|---|---|---|
| Air | gas, 0 °C, 1 atm | 1.00059 |
| Methane, $CH_4$ | gas, 0 °C, 1 atm | 1.00088 |
| Hydrogen chloride, HCl | gas, 0 °C, 1 atm | 1.0046 |
| Water, $H_2O$ | gas, 110 °C, 1 atm | 1.0126 |
| | liquid, 20 °C | 80.4 |
| Benzene, $C_6H_6$ | liquid, 20 °C | 2.28 |
| Methanol, $CH_3OH$ | liquid, 20 °C | 33.6 |
| Ammonia, $NH_3$ | liquid, −34 °C | 22.6 |
| Mineral oil | liquid, 20 °C | 2.24 |
| Sodium chloride, NaCl | solid, 20 °C | 6.12 |
| Sulfur, S | solid, 20 °C | 4.0 |
| Silicon, Si | solid, 20 °C | 11.7 |
| Polyethylene | solid, 20 °C | 2.25–2.3 |
| Porcelain | solid, 20 °C | 6.0-8.0 |
| Paraffin wax | solid, 20 °C | 2.1–2.5 |
| Pyrex glass 7070 | solid, 20 °C | 4.00 |

# Atomic Polarizability



The time-average distribution in the normal hydrogen atom. Shading represents density of electronic (negative) charge.




In an electric field, the negative charge is pulled one way and the positive nucleus is pulled the other way. The distortion is grossly exaggerated in this picture. To distort the atom that much would require a field of $10^{10}$ volts/m.

# Atomic Polarizability

Since the atom was spherically symmetrical before the field **E** was applied, the dipole moment vector **p** will be in the direction of **E**. The factor that relates **p** to **E** is called the *atomic polarizability*, and is usually denoted by $\alpha$:

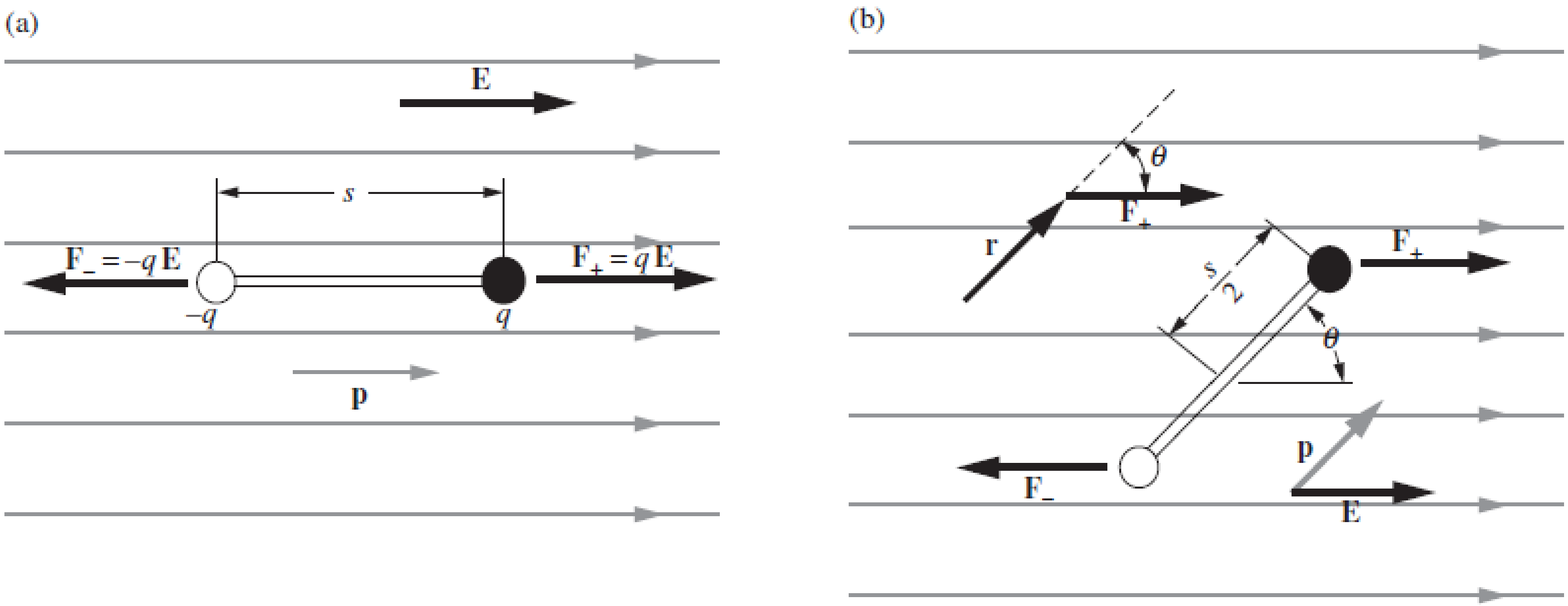
$$\mathbf{p} = \alpha \mathbf{E}$$

We should expect the relative distortion of the atom's structure, measured by the ratio $\Delta z/a$, to have the same order of magnitude as the ratio of the perturbing field $E$ to the internal fields that hold the atom together. We predict, in other words, that

$$\frac{\Delta z}{a} \approx \frac{E}{e/4\pi\epsilon_0 a^2}$$

$$p = e\,\Delta z \approx 4\pi\epsilon_0 a^3 E$$

# Torque & Force on an dipole



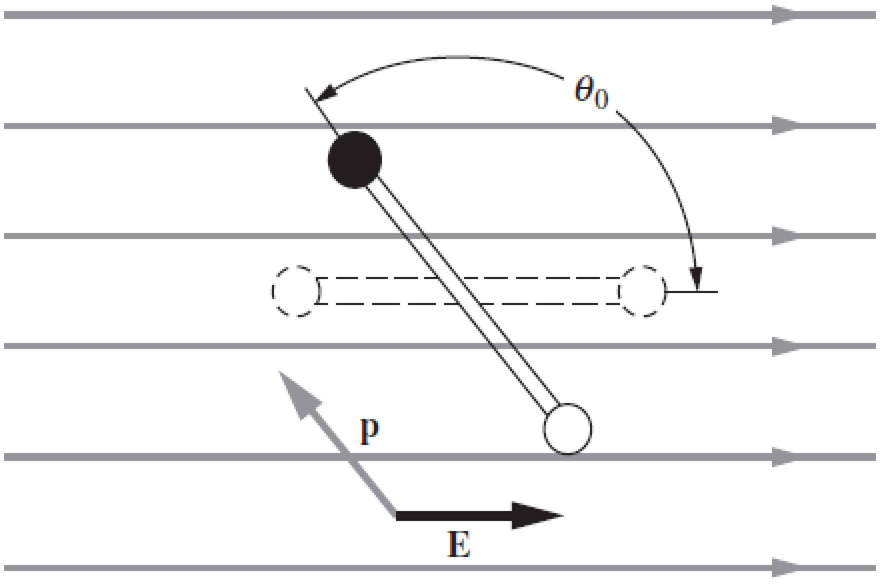

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}_{+} + (-\mathbf{r}) \times \mathbf{F}_{-}.$$

$$N = \frac{s}{2} qE \sin\theta + \frac{s}{2} qE \sin\theta = sqE \sin\theta = pE \sin\theta$$

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}$$

When the total force on the dipole is zero, as it is in this case, the torque is independent of the choice of origin

# Work required to rotate dipole



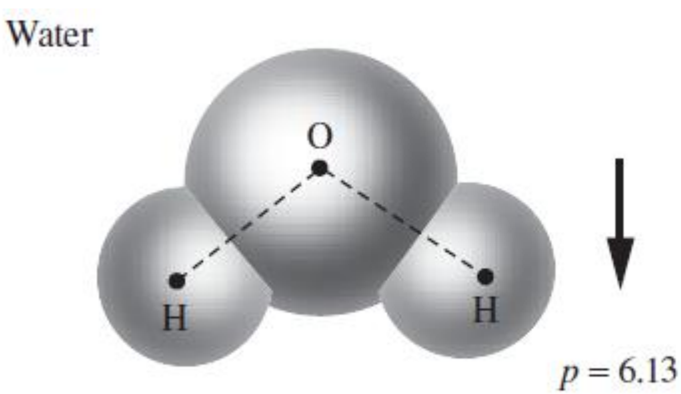

$$\int_0^{\theta_0} N \, d\theta = \int_0^{\theta_0} pE \sin\theta \, d\theta = pE(1 - \cos\theta_0)$$

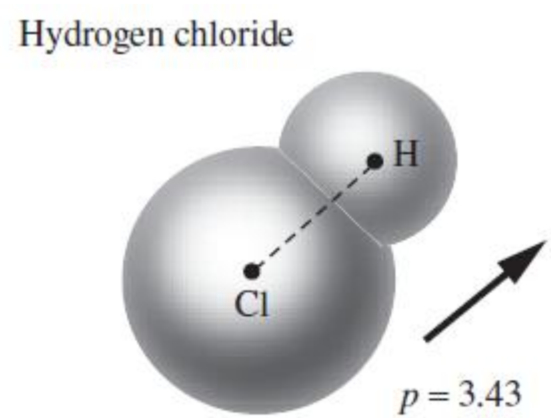
# Force on a dipole (Non-uniform E)

$$\mathbf{F} = \mathbf{F}_+ + \mathbf{F}_- = q(\mathbf{E}_+ - \mathbf{E}_-) = q(\Delta\mathbf{E})$$

$$\Delta E_x \equiv (\nabla E_x) \cdot \mathbf{d}, \quad \Longrightarrow \quad \Delta\mathbf{E} = (\mathbf{d} \cdot \nabla)\mathbf{E}$$

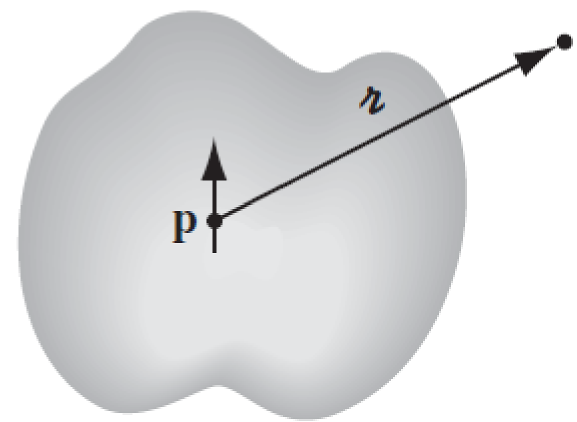
$$\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}.$$



# The Field of a Polarized Object

For a single dipole **p** ⟹ $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$




$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$$ ⟹ total potential

$$V = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \mathbf{P}\cdot\nabla'\left(\frac{1}{\mathscr{r}}\right)d\tau'$$

$$\nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$$

$$V = \frac{1}{4\pi\epsilon_0}\left[\int_{\mathcal{V}} \nabla'\cdot\left(\frac{\mathbf{P}}{\mathscr{r}}\right)d\tau' - \int_{\mathcal{V}} \frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{P})\,d\tau'\right]$$

# The Field of a Polarized Object

$$V = \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{1}{\imath} \mathbf{P} \cdot d\mathbf{a}' - \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{1}{\imath} (\nabla' \cdot \mathbf{P}) \, d\tau'$$

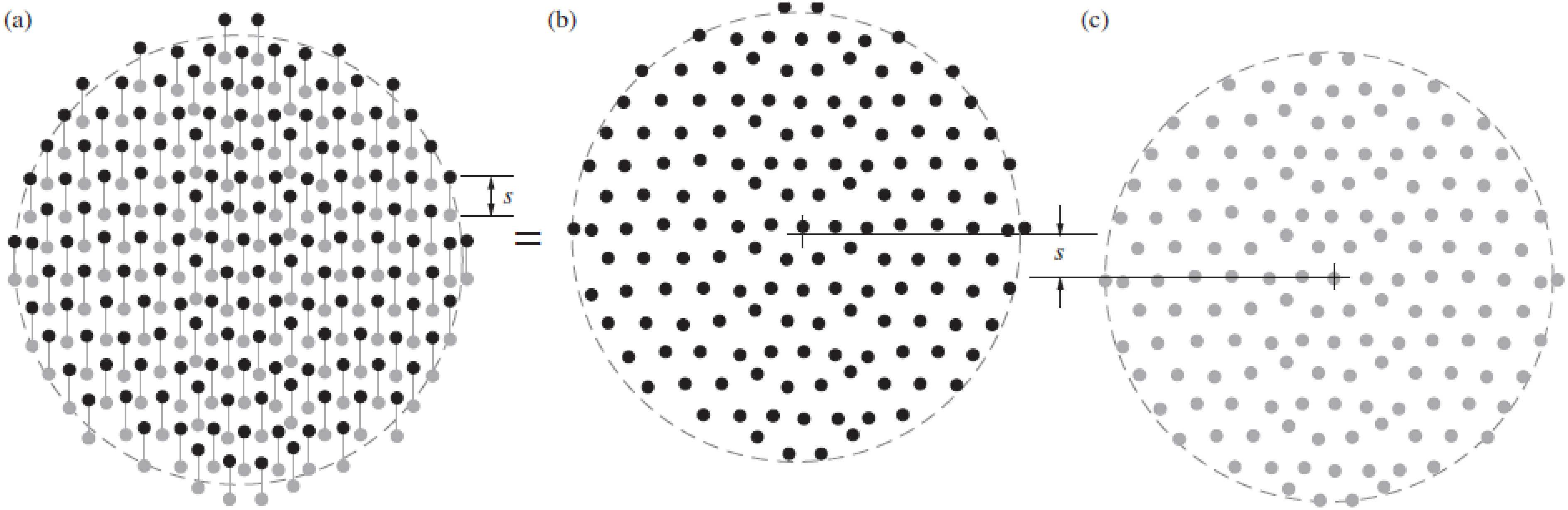
⇩

divergence theorem,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \frac{\sigma_b}{\imath} \, da' + \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho_b}{\imath} \, d\tau'$$

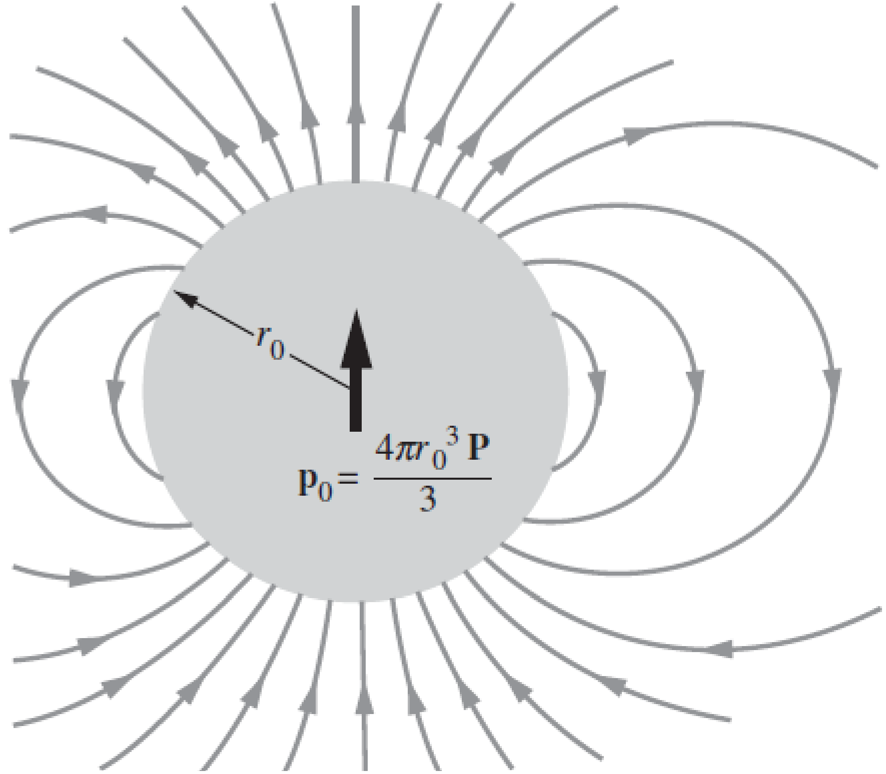
$$\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}} \qquad \rho_b \equiv -\nabla \cdot \mathbf{P}.$$

# Electric field of a Uniform Polarized Sphere



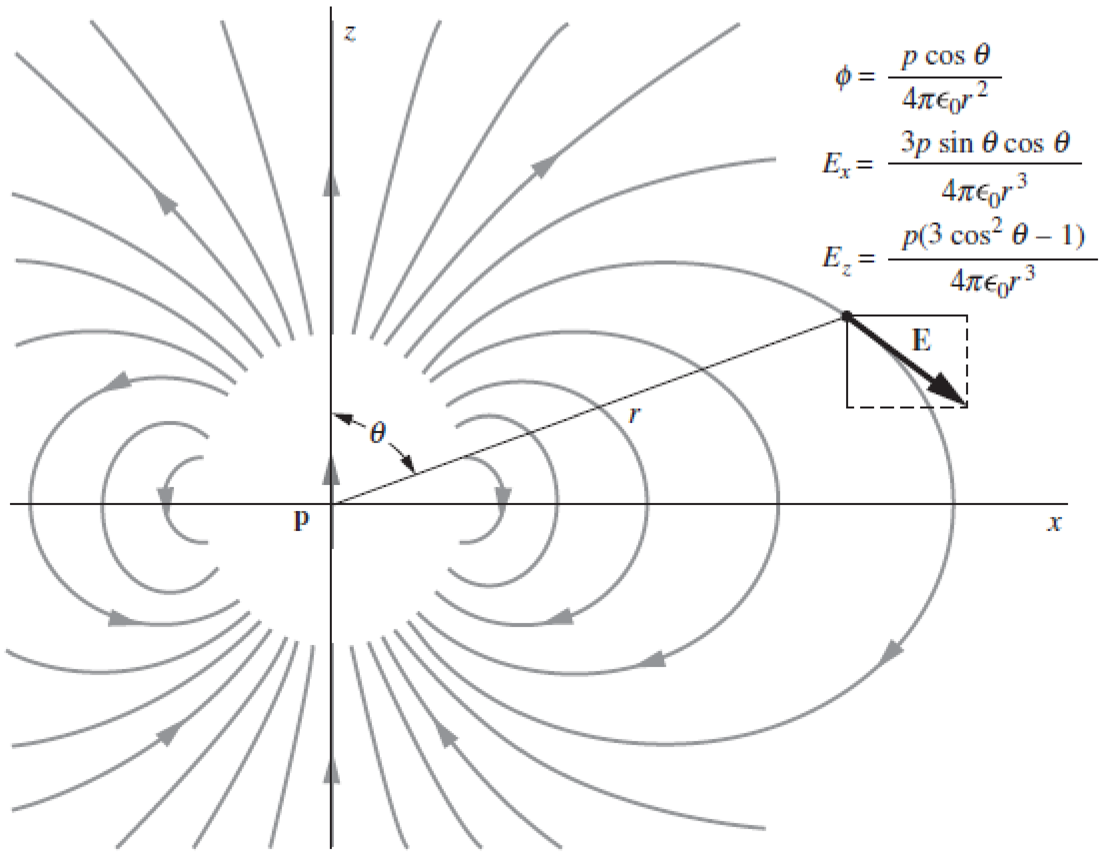

A sphere of lined-up molecular dipoles (a) is equivalent to superposed, slightly displaced, spheres of positive (b) and negative (c) charges.

# Electric field of a Uniform Polarized Sphere

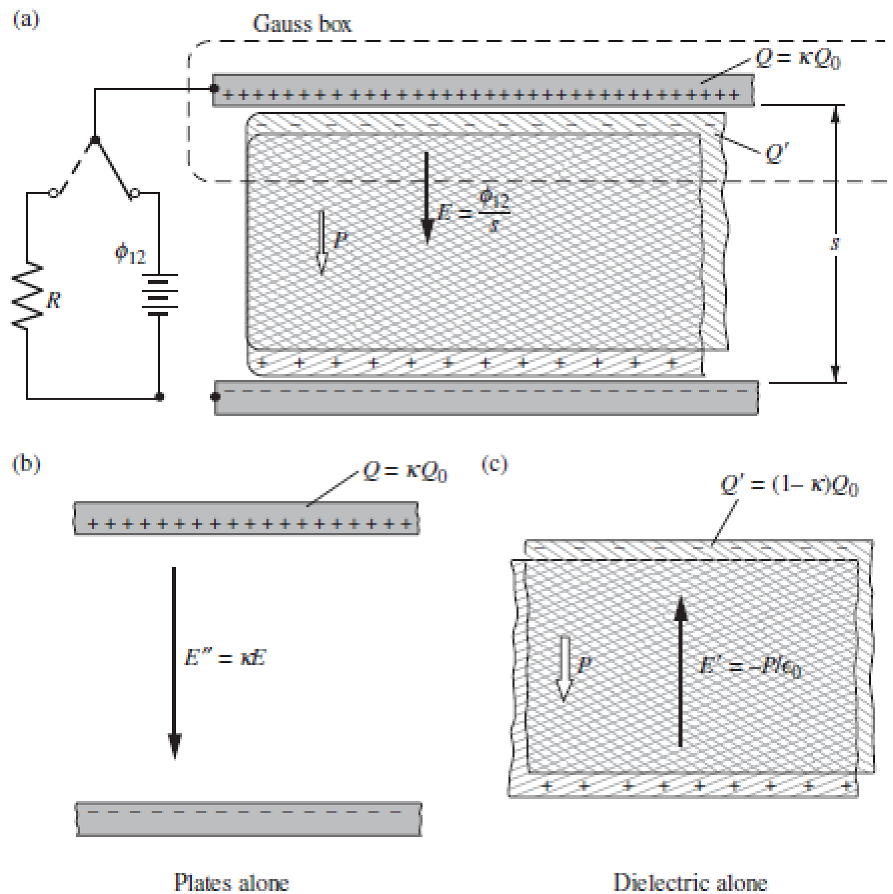



The field outside a uniformly polarized sphere is exactly the same as that of a dipole located at the center of the sphere.

# Electric Field of a Dipole



# Electric Field inside a Capacitor



# Electric Field inside a Capacitor

The fact that the **E** fields are the same implies that the total charge on and near the top plate in the dielectric-filled capacitor must be the same as it was in the empty capacitor, namely $Q_0$. Now, the charge is made up of two parts, the charge on the plate $Q$ (which will flow off when the capacitor is discharged) and $Q'$, the charge that belongs to the dielectric. The charge on the plate is given by $Q = \kappa Q_0$. That was our definition of $\kappa$. Therefore, if $Q + Q' = Q_0$ as we have just concluded, we must have

$$Q' = Q_0 - Q = Q_0(1 - \kappa)$$

We can think of this system as the superposition of a vacuum capacitor and a polarized dielectric slab. . In the vacuum capacitor with charge $\kappa Q_0$, the electric field $E''$ would be $\kappa$ times the field $E$. In the isolated polarized dielectric slab the field $E'$ is $-P/\epsilon_0$, as superposition of these two objects creates the actual field $E$.

$$E = E'' + E' = \kappa E - \frac{P}{\epsilon_0}$$

# Electric Field inside a Capacitor

$$E = E'' + E' = \kappa E - \frac{P}{\epsilon_0}$$

which can be rearranged like this:

$$\frac{P}{\epsilon_0 E} = \kappa - 1$$

The ratio $P/\epsilon_0 E$ (which is dimensionless) is called the electric susceptibility of the dielectric material and is denoted by $\chi_e$ (Greek chi):

$$\chi_e \equiv \frac{P}{\epsilon_0 E} \quad \Longrightarrow \quad \boxed{P = \chi_e \epsilon_0 E}$$

$$\chi_e = \kappa - 1 \quad \Longrightarrow \quad \boxed{\kappa = 1 + \chi_e}$$

# Gauss's Law in presence of Dielectric

$$\rho = \rho_b + \rho_f$$

(bound) charge, $\rho_b = -\nabla \cdot \mathbf{P}$ within the dielectric and $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$ on the surface.

and Gauss's law reads

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f$$

where $\mathbf{E}$ is now the *total* field, not just that portion generated by polarization.

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f .$$

$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P},$ is known as the **electric displacement**.

In terms of $\mathbf{D}$, Gauss's law reads

$$\nabla \cdot \mathbf{D} = \rho_f .$$

# Gauss's Law in presence of Dielectric

In terms of **D**, Gauss's law reads

in integral form,

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$$

# Boundary Conditions

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f,$$

$$\mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}}$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma$$

$$\mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = \mathbf{0}$$

# *Linear Dielectrics*

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

The constant of proportionality, $\chi_e$, is called the **electric susceptibility** of the medium (a factor of $\epsilon_0$ has been extracted to make $\chi_e$ dimensionless).

In linear media we have

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E},$$

so **D** is *also* proportional to **E**:

$$\mathbf{D} = \epsilon \mathbf{E} \qquad \epsilon \equiv \epsilon_0 (1 + \chi_e)$$

This new constant $\epsilon$ is called the **permittivity** of the material.

$\epsilon_0$ is called the **permittivity of free space**

# Boundary Conditions (Linear Dielectrics)

$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left( \epsilon_0 \frac{\chi_e}{\epsilon} \mathbf{D} \right) = -\left( \frac{\chi_e}{1 + \chi_e} \right) \rho_f .$$

$$\epsilon_{\text{above}} E_{\text{above}}^{\perp} - \epsilon_{\text{below}} E_{\text{below}}^{\perp} = \sigma_f$$

$$\epsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \epsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f$$

$$V_{\text{above}} = V_{\text{below}}$$

# Energy in Dielectric System

$$W = \tfrac{1}{2}CV^2$$ work to charge up a capacitor

If the capacitor is filled with linear dielectric, its capacitance exceeds the vacuum value by a factor of the dielectric constant,

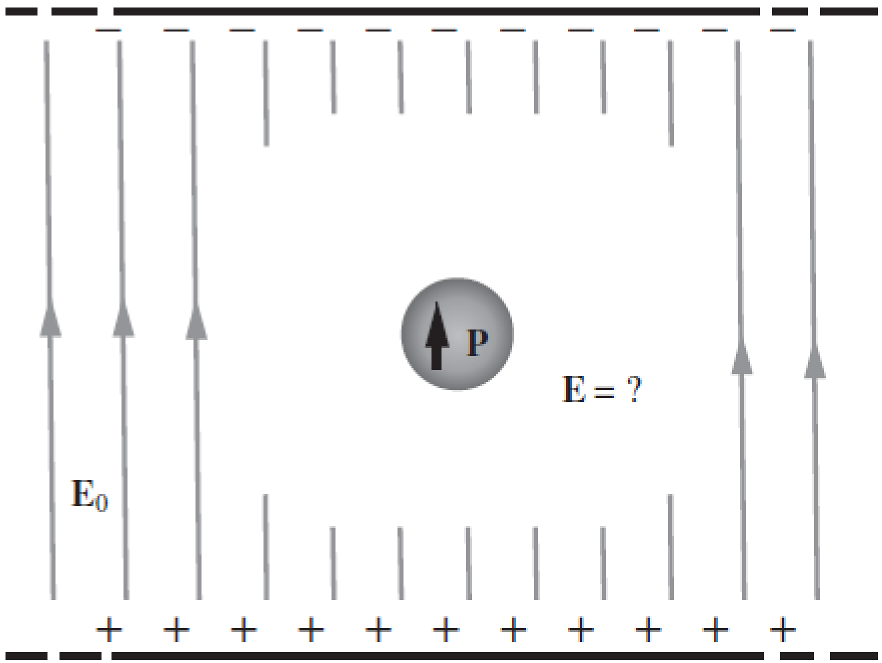
$$C = \epsilon_r C_{\text{vac}}$$

$$W = \frac{\epsilon_0}{2}\int E^2\, d\tau$$ general formula for the energy stored

$$W = \frac{\epsilon_0}{2}\int \epsilon_r E^2\, d\tau = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, d\tau$$

# A Dielectric Sphere in a Uniform Field

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}'.$$




The sources of the field $\mathbf{E}_0$ remain fixed. The dielectric sphere develops some polarization **P**. The total field **E** is the superposition of $\mathbf{E}_0$ and the field of this polarized sphere.

The total field **E** is no longer uniform in the neighborhood of the sphere. It is the *sum* of the uniform field $\mathbf{E}_0$ of the distant sources and a field $\mathbf{E}'$ generated by the polarized matter itself:

# A Dielectric Sphere in a Uniform Field

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}'.$$

This relation is valid both inside and outside the sphere. The field $\mathbf{E}'$ depends on the polarization $\mathbf{P}$ of the dielectric, which in turn depends on the value of $\mathbf{E}$ inside the sphere:

$$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E}_{\text{in}} = (\kappa - 1)\epsilon_0 \mathbf{E}_{\text{in}}.$$

Remember that the $\mathbf{E}$ that appears in this expression involving $\chi_e$ is the *total* electric field.

$$\mathbf{E}'_{\text{in}} = -\frac{\mathbf{P}}{3\epsilon_0}.$$

relation between the polarization $\mathbf{P}$ of the sphere and its own field at points inside, $\mathbf{E}'_{\text{in}}$

$$\mathbf{E}'_{\text{in}} = -(\kappa - 1)\mathbf{E}_{\text{in}}/3.$$

# A Dielectric Sphere in a Uniform Field

$$\mathbf{E}_{\text{in}} = \mathbf{E}_0 - \frac{\kappa - 1}{3}\mathbf{E}_{\text{in}}$$

$$\mathbf{E}_{\text{in}} = \left(\frac{3}{2+\kappa}\right)\mathbf{E}_0$$

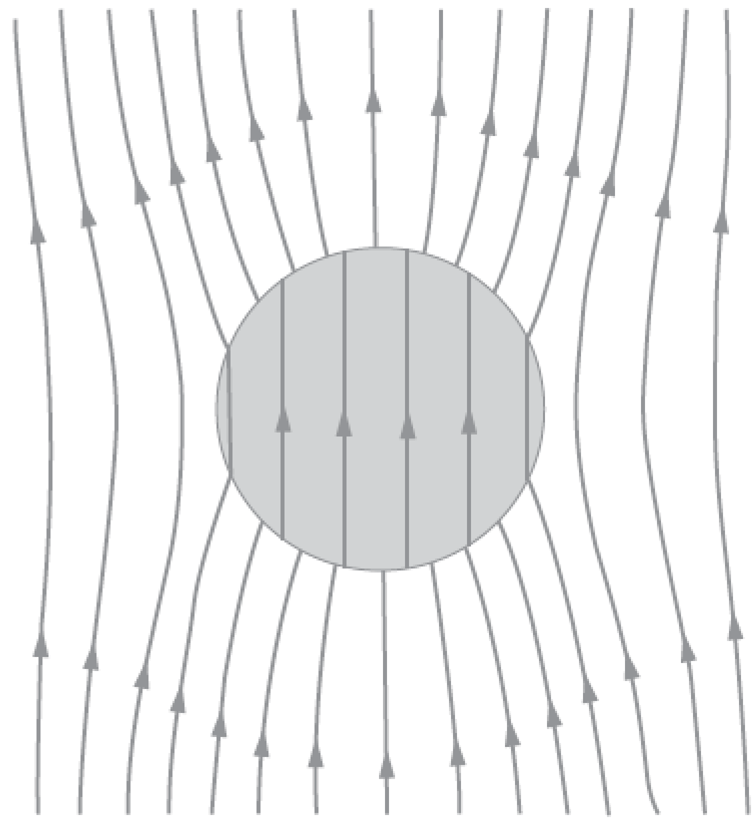
total field inside the sphere as

Because $\kappa$ is greater than 1, the factor $3/(2+\kappa)$ will be less than 1; the field inside the dielectric is weaker than $\mathbf{E}_0$. The polarization is

$$\mathbf{P} = (\kappa - 1)\epsilon_0\mathbf{E}_{\text{in}}$$

$$\mathbf{P} = 3\left(\frac{\kappa - 1}{\kappa + 2}\right)\epsilon_0\mathbf{E}_0$$

# A Dielectric Sphere in a Uniform Field



The total field $\mathbf{E}$, both inside and outside the dielectric sphere.

# A Dielectric Sphere in a Uniform Field

To summarize, we found $\mathbf{E}_{\text{in}}$ by effectively equating two different expressions for the field $\mathbf{E}'_{\text{in}}$ caused by the polarized matter. One expression is simply the statement of superposition, $\mathbf{E}'_{\text{in}} = \mathbf{E}_{\text{in}} - \mathbf{E}_0$. The other expression is $\mathbf{E}'_{\text{in}} = -(\kappa - 1)\mathbf{E}_{\text{in}}/3$, which comes from the facts that $\mathbf{E}'_{\text{in}}$ is proportional to $\mathbf{P}$ (in the case of a sphere) and that $\mathbf{P}$ is proportional to $\mathbf{E}_{\text{in}}$ (in a linear dielectric).

To compute the total field $\mathbf{E}_{\text{out}}$ outside the sphere we must add vectorially to $\mathbf{E}_0$ the field of a central dipole with dipole moment equal to $\mathbf{P}$ times the volume of the sphere.

# *Clausius-Mossotti Formula for a nonpolar substance*

*Question:*

What is the relation between the atomic polarizability $\alpha$ and the susceptibility $\chi_e$?

In a linear dielectric ⇨ $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$ polarization is proportional to the field

If the material consists of atoms (or nonpolar molecules), the induced dipole moment of each one is likewise proportional to the field $\mathbf{p} = \alpha \mathbf{E}$

Since **P** (the dipole moment per unit volume) is **p** (the dipole moment per atom) times $N$ (the number of atoms per unit volume), $\mathbf{P} = N\mathbf{p} = N\alpha\mathbf{E}$, one's first inclination is to say that

$$\chi_e = \frac{N\alpha}{\epsilon_0}$$

**True if Density is Low**

# Clausius-Mossotti Formula

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$ → Total Macroscopic field in the medium

$$\mathbf{p} = \alpha \mathbf{E}$$ → Field due to everything except the particular atom under consideration

Imagine that the space allotted to each atom is a sphere of radius $R$

The density of atoms is $N = \frac{1}{(4/3)\pi R^3}$.

# Clausius-Mossotti Formula

The macroscopic field $\mathbf{E}$ is $\mathbf{E}_{\text{self}} + \mathbf{E}_{\text{else}}$

$\mathbf{E}_{\text{self}}$ is the average field over the sphere due to the atom itself.

$$\mathbf{p} = \alpha \mathbf{E}_{\text{else}} \;\Rightarrow\; \mathbf{P} = N\alpha \mathbf{E}_{\text{else}}$$

$$\mathbf{E}_{\text{self}} = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}$$

$$\begin{aligned}\mathbf{E} &= -\frac{1}{4\pi\epsilon_0}\frac{\alpha}{R^3}\mathbf{E}_{\text{else}} + \mathbf{E}_{\text{else}} \\ &= \left(1 - \frac{\alpha}{4\pi\epsilon_0 R^3}\right)\mathbf{E}_{\text{else}} = \left(1 - \frac{N\alpha}{3\epsilon_0}\right)\mathbf{E}_{\text{else}}\end{aligned}$$

# Clausius-Mossotti Formula

$$\mathbf{E} = -\frac{1}{4\pi\epsilon_0}\frac{\alpha}{R^3}\mathbf{E}_{\text{else}} + \mathbf{E}_{\text{else}}$$

$$= \left(1 - \frac{N\alpha}{3\epsilon_0}\right)\mathbf{E}_{\text{else}}$$

$$\mathbf{P} = N\alpha\mathbf{E}_{\text{else}}$$

$$\mathbf{P} = \frac{N\alpha}{(1 - N\alpha/3\epsilon_0)}\mathbf{E} = \epsilon_0\chi_e\mathbf{E}$$

$$\chi_e = \frac{N\alpha/\epsilon_0}{(1 - N\alpha/3\epsilon_0)}$$

# Clausius-Mossotti Formula

$$\chi_e = \frac{N\alpha/\epsilon_0}{(1 - N\alpha/3\epsilon_0)} \quad \Rightarrow \quad \chi_e - \frac{N\alpha}{3\epsilon_0}\chi_e = \frac{N\alpha}{\epsilon_0}$$

$$\frac{N\alpha}{\epsilon_0}\left(1 + \frac{\chi_e}{3}\right) = \chi_e$$

$$\alpha = \frac{\epsilon_0}{N}\frac{\chi_e}{(1 + \chi_e/3)} = \frac{3\epsilon_0}{N}\frac{\chi_e}{(3 + \chi_e}$$

$$\chi_e = \epsilon_r - 1$$

$$\alpha = \frac{3\epsilon_0}{N}\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)$$

# *Langevin Formula for a polar substance*

The **Langevin equation** tells you how to calculate the susceptibility of a *polar* substance, in terms of the permanent molecular dipole moment $p$

The energy of a dipole in an external field $\mathbf{E}$ is $u = -\mathbf{p} \cdot \mathbf{E} = -pE\cos\theta$ where $\theta$ is the usual polar angle, if we orient the $z$ axis along $\mathbf{E}$.

Statistical mechanics says that for a material in equilibrium at absolute temperature $T$, the probability of a given molecule having energy $u$ is proportional to the Boltzmann factor,

$$\exp(-u/kT)$$

# Langevin Formula

The average energy of the dipoles is therefore

$$<u> = \frac{\displaystyle\int u e^{-(u/kT)}\, d\Omega}{\displaystyle\int e^{-(u/kT)}\, d\Omega} \qquad \text{where } \ d\Omega = \sin\theta\, d\theta\, d\phi$$

$$\langle u \rangle = \frac{\int_{-pE}^{pE} u e^{-u/kT}\, du}{\int_{-pE}^{pE} e^{-u/kT}\, du} = \frac{(kT)^2 e^{-u/kT}\, [-(u/kT) - 1]|_{-pE}^{pE}}{-kT e^{-u/kT}|_{-pE}^{pE}}$$

$$= kT \left\{ \frac{\left[e^{-pE/kT} - e^{pE/kT}\right] + \left[(pE/kT)e^{-pE/kT} + (pE/kT)e^{pE/kT}\right]}{e^{-pE/kT} - e^{pE/kT}} \right\}$$

# Langevin Formula

$$\langle u \rangle = kT - pE \left[ \frac{e^{pE/kT} + e^{-pE/kT}}{e^{pE/kT} - e^{-pE/kT}} \right] = kT - pE \coth\left( \frac{pE}{kT} \right)$$

$$\mathbf{P} = N\langle \mathbf{p} \rangle$$

$$\langle \mathbf{p} \rangle = \langle p \cos\theta \rangle \hat{\mathbf{E}}$$

$$= \langle \mathbf{p} \cdot \mathbf{E} \rangle (\hat{\mathbf{E}}/E) = -\langle u \rangle (\hat{\mathbf{E}}/E)$$

$$P = Np \frac{-\langle u \rangle}{pE}$$

$$P = Np \left\{ \coth\left( \frac{pE}{kT} \right) - \frac{kT}{pE} \right\}$$

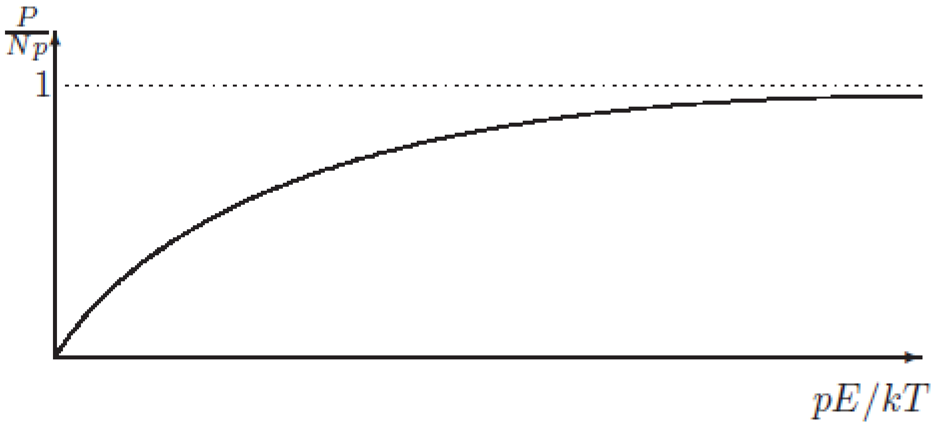
# Langevin Formula

Let $y \equiv P/Np, \; x \equiv pE/kT$

Then $y = \coth x - 1/x$.

As $x \to 0$, $y = \left(\frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \cdots\right) - \frac{1}{x} = \frac{x}{3} - \frac{x^3}{45} + \cdots$

As $x \to \infty$, $y \to \coth(\infty) = 1$



# Langevin Formula

For small $x$, $y \approx \frac{1}{3}x$

$$\frac{P}{Np} \approx \frac{pE}{3kT}$$

$$P \approx \frac{Np^2}{3kT}E = \epsilon_0 \chi_e E$$

$$\chi_e = \frac{Np^2}{3\epsilon_0 kT}$$

# Average potential (Laplace's Equation)

Electrostatic potential at any point P in a charge-free region is equal to its average value over any spherical surface ( radius R) centered at P

$$V_{\text{ave}}(R) = \frac{1}{4\pi R^2}\int V(\mathbf{r})\, da,$$

$$V_{\text{ave}}(R) = \frac{1}{4\pi}\int V(R,\theta,\phi)\sin\theta\, d\theta\, d\phi.$$

# Average potential (Laplace's Equation)

$$V_{\text{ave}}(R) = \frac{1}{4\pi}\int V(R,\theta,\phi)\sin\theta\, d\theta\, d\phi.$$

$$\frac{dV_{\text{ave}}}{dR} = \frac{1}{4\pi}\int \frac{\partial V}{\partial R}\sin\theta\, d\theta\, d\phi = \frac{1}{4\pi}\int (\nabla V\cdot\hat{\mathbf{r}})\sin\theta\, d\theta\, d\phi$$

$$= \frac{1}{4\pi R^2}\int (\nabla V)\cdot(R^2\sin\theta\, d\theta\, d\phi\,\hat{\mathbf{r}})$$

$$= \frac{1}{4\pi R^2}\int (\nabla V)\cdot d\mathbf{a}$$

$$= \frac{1}{4\pi R^2}\int (\nabla^2 V)\, d\tau$$

$$= 0$$

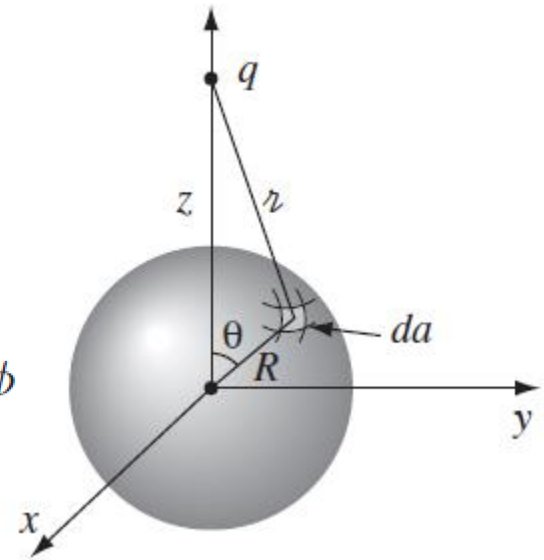
So $V_{\text{ave}}$ is independent of $R$—the same for all spheres

$$\boxed{V_{\text{ave}}(R) = V(0)} \qquad \text{(taking the limit as } R\to 0)$$

# Average potential (Coulomb's Law)

The value of $V$ at point $\mathbf{r}$ is the average value of $V$ over a spherical surface of radius $R$ centered at $\mathbf{r}$:

$$V(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V\, da.$$




$$\mathscr{r}^2 = z^2 + R^2 - 2zR\cos\theta$$

$$V_{\text{ave}} = \frac{1}{4\pi R^2} \frac{q}{4\pi\epsilon_0} \int [z^2 + R^2 - 2zR\cos\theta]^{-1/2} R^2 \sin\theta\, d\theta\, d\phi$$

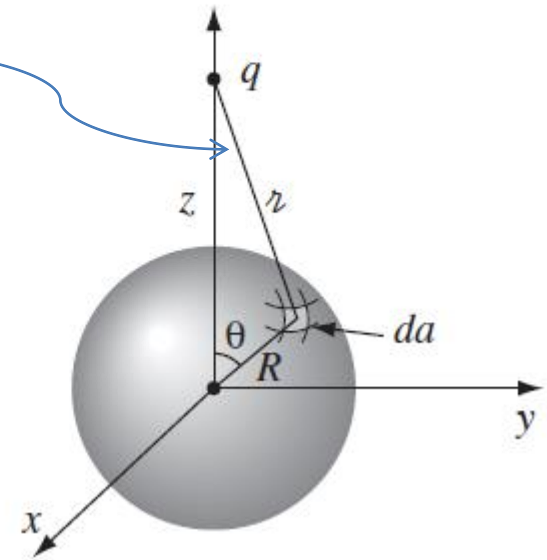
$$= \frac{q}{4\pi\epsilon_0} \frac{1}{2zR} \sqrt{z^2 + R^2 - 2zR\cos\theta}\,\Big|_0^\pi$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{2zR} [(z+R) - (z-R)]$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{z}$$ ➡ potential due to $q$ at the *center* of the sphere

# Average Electric field (Coulomb's Law)

$\alpha$ be the angle between $\mathscr{r}$ and the $z$ axis




$$\mathbf{E}_{\text{ave}} = \frac{1}{4\pi R^2} \oint \mathbf{E}\, da$$

$$= -\hat{\mathbf{z}} \frac{1}{4\pi R^2} \frac{q}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}^2} \cos\alpha\, da$$

$\mathbf{E}_{\text{ave}}$ points in the $-\hat{\mathbf{z}}$

$$R^2 = z^2 + \mathscr{r}^2 - 2\mathscr{r}\, z \cos\alpha$$

$$\cos\alpha = \frac{z^2 + \mathscr{r}^2 - R^2}{2\mathscr{r}\, z}$$

# Average Electric field (Coulomb's Law)

$$\cos\alpha = \frac{z^2 + \mathscr{r}^2 - R^2}{2\mathscr{r}\, z} \;\Rightarrow\; \frac{\cos\alpha}{\mathscr{r}^2} = \frac{z^2 + \mathscr{r}^2 - R^2}{2z\mathscr{r}^3}$$

$$\mathscr{r}^2 = R^2 + z^2 - 2Rz\cos\theta \qquad = \frac{z - R\cos\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}$$

$$\mathbf{E}_{\text{ave}} = -\hat{\mathbf{z}}\frac{q}{16\pi^2 R^2\epsilon_0}\int \frac{z - R\cos\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}} R^2 \sin\theta \, d\theta \, d\phi$$

$$= -\frac{q\hat{\mathbf{z}}}{8\pi\epsilon_0}\int_0^\pi \frac{z - R\cos\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}} \sin\theta \, d\theta$$

$$= -\frac{q\hat{\mathbf{z}}}{8\pi\epsilon_0}\int_{-1}^1 \frac{z - Ru}{(R^2 + z^2 - 2Rzu)^{3/2}} \, du$$

The integral $I$ where $u \equiv \cos\theta$

# Average Electric field (Coulomb's Law)

$$I = \frac{1}{R\sqrt{R^2+z^2-2Rzu}}\Bigg|_{-1}^{1} - \frac{1}{2Rz^2}\left(\sqrt{R^2+z^2-2Rzu} + \frac{R^2+z^2}{\sqrt{R^2+z^2-2Rzu}}\right)\Bigg|_{-1}^{1}$$

$$= \frac{1}{R}\left(\frac{1}{|z-R|} - \frac{1}{z+R}\right) - \frac{1}{2Rz^2}\left[|z-R| - (z+R) + (R^2+z^2)\left(\frac{1}{|z-R|} - \frac{1}{z+R}\right)\right]$$

(a) If $z > R$,

$$I = \frac{1}{R}\left(\frac{1}{z-R} - \frac{1}{z+R}\right) - \frac{1}{2Rz^2}\left[(z-R) - (z+R) + (R^2+z^2)\left(\frac{1}{z-R} - \frac{1}{z+R}\right)\right]$$

$$= \frac{1}{R}\left(\frac{2R}{z^2-R^2}\right) - \frac{1}{2Rz^2}\left[-2R + (R^2+z^2)\frac{2R}{z^2-R^2}\right] = \frac{2}{z^2}.$$

$$\mathbf{E}_{\text{ave}} = -\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\,\hat{\mathbf{z}}$$

# Average Electric field (Coulomb's Law)

$$I = \frac{1}{R\sqrt{R^2+z^2-2Rzu}}\Bigg|_{-1}^{1} - \frac{1}{2Rz^2}\left(\sqrt{R^2+z^2-2Rzu} + \frac{R^2+z^2}{\sqrt{R^2+z^2-2Rzu}}\right)\Bigg|_{-1}^{1}$$

$$= \frac{1}{R}\left(\frac{1}{|z-R|} - \frac{1}{z+R}\right) - \frac{1}{2Rz^2}\left[|z-R| - (z+R) + (R^2+z^2)\left(\frac{1}{|z-R|} - \frac{1}{z+R}\right)\right]$$

(b) If $z < R$,

$$I = \frac{1}{R}\left(\frac{1}{R-z} - \frac{1}{z+R}\right) - \frac{1}{2Rz^2}\left[(R-z) - (z+R) + (R^2+z^2)\left(\frac{1}{R-z} - \frac{1}{z+R}\right)\right]$$

$$= \frac{1}{R}\left(\frac{2z}{R^2-z^2}\right) - \frac{1}{2Rz^2}\left[-2z + (R^2+z^2)\frac{2z}{R^2-z^2}\right] = 0.$$

$$\mathbf{E}_{\text{ave}} = \mathbf{0}$$

# Laplace's Equation in spherical coordinates

In the spherical system, Laplace's equation reads:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

Azimuthal Symmetry

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

$V(r,\theta)=R(r)\Theta(\theta)$ → Solution for separation of variables

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0$$

# Laplace's Equation in spherical coordinates

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0$$

first term depends only on $r$, and the second only on $\theta$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)$$

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)$$

l(l+1) separation constant

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R$$

General Solution

$$R(r) = Ar^l + \frac{B}{r^{l+1}}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\,\Theta$$

Legendre Polynomials

$$\Theta(\theta) = P_l(\cos\theta)$$

# Solution of Laplace's equation for spherical symmetry

General solution is the linear combination of all solutions

$$V(r,\theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta).$$

## CASE -1(Potential inside the sphere)

The potential $V_0(\theta)$ is specified on the surface of a hollow sphere

$B_l = 0$ for all $l$—otherwise the potential would blow up at the origin

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

At $r = R$ this must match the specified function $V_0(\theta)$:

$$V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta)$$

## Fourier analysis to determine the constants

$$V(R,\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = V_0(\theta)$$

Multiplying by

$P_{l'}(\cos\theta)\sin\theta$

and integrating

$$\int_{-1}^{1} P_l(x)P_{l'}(x)\,dx = \int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\,d\theta$$

$$= \begin{cases} 0, & \text{if } l' \neq l, \\ \dfrac{2}{2l+1}, & \text{if } l' = l. \end{cases}$$

$$A_{l'} R^{l'} \frac{2}{2l'+1} = \int_0^{\pi} V_0(\theta) P_{l'}(\cos\theta)\sin\theta\,d\theta.$$

$$A_l = \frac{2l+1}{2R^l}\int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\,d\theta$$

# Potential is determined outside the sphere

CASE-2 (Potential specified on the surface but to be determined for r > R)

In this case it's the $A_l$'s that must be zero

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

At the surface of the sphere,

$$V(R,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0(\theta)$$

$$\frac{B_{l'}}{R^{l'+1}} \frac{2}{2l'+1} = \int_0^{\pi} V_0(\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta$$

Using orthogonality of Legendre Polynomial

$$B_l = \frac{2l+1}{2} R^{l+1} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta$$

Constant Potential on the surface of the sphere. Find V, inside and outside the sphere

Suppose the potential is a *constant* $V_0$ over the surface of the sphere.

$$\textit{Inside:}\ V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$A_l = \frac{(2l+1)}{2R^l} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta$$

$$P_0(\cos\theta) = 1$$

Insert in the integral

$$A_l = \frac{(2l+1)V_0}{2R^l} \int_0^{\pi} P_l(\cos\theta) \sin\theta \, d\theta$$

$$\int_0^{\pi} P_0(\cos\theta) P_l(\cos\theta) \sin\theta \, d\theta = \begin{Bmatrix} 0, & \text{if } l \neq 0 \\ 2, & \text{if } l = 0 \end{Bmatrix}$$

# Potential inside the sphere

$$A_l = \frac{(2l+1)V_0}{2R^l} \int_0^\pi P_l(\cos\theta)\sin\theta \, d\theta$$

$$A_l = \left\{ \begin{matrix} 0, & \text{if } l \neq 0 \\ V_0, & \text{if } l = 0 \end{matrix} \right\}$$

$$V(r,\theta) = A_0 \, r^0 P_0(\cos\theta) = V_0$$

The potential is *constant throughout the sphere.*

# Potential outside the sphere

$$\textit{Outside:}\ V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$B_l = \frac{(2l+1)}{2} R^{l+1} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta$$

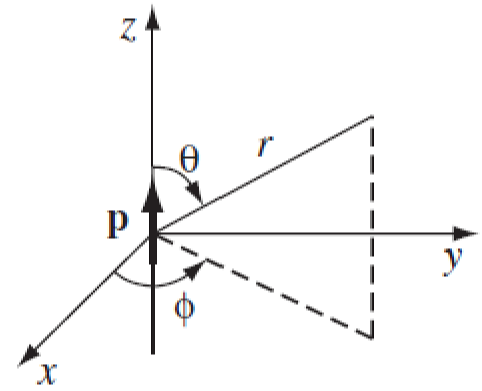
$$= \frac{(2l+1)}{2} R^{l+1} V_0 \int_0^{\pi} P_l(\cos\theta) \sin\theta \, d\theta = \left\{ \begin{array}{ll} 0, & \text{if } l \neq 0 \\ RV_0, & \text{if } l = 0 \end{array} \right\}$$

$$V(r,\theta) = V_0 \frac{R}{r}$$

equals $V_0$ at $r = R$, then falls off like $\frac{1}{r}$

# Dipole Electric field

$$V_{dip}(r,\theta) = \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}.$$



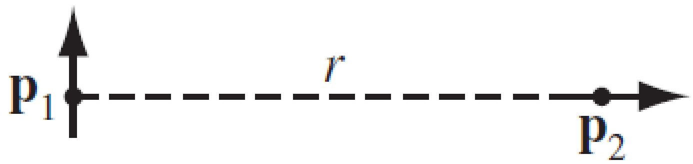

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = 0$$

$$\mathbf{E}_{dip}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$$

# Torque on dipoles due to their mutual interaction




Field of $\mathbf{p}_1$ at $\mathbf{p}_2$ $\quad \mathbf{E}_1 = \dfrac{p_1}{4\pi\epsilon_0 r^3}\,\hat{\boldsymbol{\theta}}$ (points *down*) $\quad \theta = \pi/2$

Torque on $\mathbf{p}_2$: $\mathbf{N}_2 = \mathbf{p}_2 \times \mathbf{E}_1$ (points *into* the page)

$$= p_2 E_1 \sin 90^\circ = p_2 E_1$$

$$= \frac{p_1 p_2}{4\pi\epsilon_0 r^3}$$

Field of $\mathbf{p}_2$ at $\mathbf{p}_1$ $\quad \mathbf{E}_2 = \dfrac{p_2}{4\pi\epsilon_0 r^3}\,(-2\,\hat{\mathbf{r}})$ (points to the *right*)

$\theta = \pi$

Torque on $\mathbf{p}_1$: $\mathbf{N}_1 = \mathbf{p}_1 \times \mathbf{E}_2$ (points *into* the page)

$$= \frac{2 p_1 p_2}{4\pi\epsilon_0 r^3}$$

# Energy of an ideal dipole

For a physical dipole, with $-q$ at $\mathbf{r}$ and $+q$ at $\mathbf{r}+\mathbf{d}$,

$$U = qV(\mathbf{r}+\mathbf{d}) - qV(\mathbf{r}) = q[V(\mathbf{r}+\mathbf{d}) - V(\mathbf{r})] = q\left[-\int_{\mathbf{r}}^{\mathbf{r}+\mathbf{d}} \mathbf{E}\cdot d\mathbf{l}\right]$$

For an ideal dipole the integral reduces to $\mathbf{E}\cdot\mathbf{d}$, and

$$\boxed{U = -q\mathbf{E}\cdot\mathbf{d} = -\mathbf{p}\cdot\mathbf{E}}$$

# Electric field of a dipole (coordinate-free form)

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right]$$

Proof:

$$\mathbf{p} = (\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} + (\mathbf{p}\cdot\hat{\boldsymbol{\theta}})\,\hat{\boldsymbol{\theta}}$$

$$= p\cos\theta\,\hat{\mathbf{r}} - p\sin\theta\,\hat{\boldsymbol{\theta}}$$

$$3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p} = 3p\cos\theta\,\hat{\mathbf{r}} - p\cos\theta\,\hat{\mathbf{r}} + p\sin\theta\,\hat{\boldsymbol{\theta}}$$

$$= 2p\cos\theta\,\hat{\mathbf{r}} + p\sin\theta\,\hat{\boldsymbol{\theta}}$$

# Interaction energy of two dipoles

$$U = -\mathbf{p_1}\cdot\mathbf{E_2}$$

$$\mathbf{E_2} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3\left(\mathbf{p_2}\cdot\hat{\mathbf{r}}\right)\hat{\mathbf{r}} - \mathbf{p_2}\right]$$

$$U = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[\mathbf{p_1}\cdot\mathbf{p_2} - 3\left(\mathbf{p_1}\cdot\hat{\mathbf{r}}\right)\left(\mathbf{p_2}\cdot\hat{\mathbf{r}}\right)\right]$$

# Potential of a uniformly polarized sphere

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\imath}}}{\imath^2}\,d\tau'$$

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}\cdot\hat{\boldsymbol{\imath}}}{\imath^2}\,d\tau$$

$$= \mathbf{P}\cdot\left\{\frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\imath}}}{\imath^2}\,d\tau\right\}$$

Field of a uniformly charged sphere divided by the volume charge density

$$\frac{1}{4\pi\epsilon_0}\int \frac{\hat{\boldsymbol{\imath}}}{\imath^2}\,d\tau = \frac{1}{\rho}\left\{\begin{array}{l} \frac{1}{4\pi\epsilon_0}\frac{(4/3)\pi R^3\rho}{r^2}\,\hat{\mathbf{r}},\ (r > R), \\ \frac{1}{4\pi\epsilon_0}\frac{(4/3)\pi R^3\rho}{R^3}\,\mathbf{r},\ (r < R). \end{array}\right\}$$

$$V(r,\theta) = \left\{\begin{array}{ll} \frac{R^3}{3\epsilon_0 r^2}\mathbf{P}\cdot\hat{\mathbf{r}} = \boxed{\frac{R^3 P\cos\theta}{3\epsilon_0 r^2},} & (r > R), \\ \frac{1}{3\epsilon_0}\mathbf{P}\cdot\mathbf{r} = \boxed{\frac{Pr\cos\theta}{3\epsilon_0},} & (r < R). \end{array}\right\}$$

A dipole p is situated at the origin , pointing in the z direction.

(a) What is the force on a point charge $q$ at $(a, 0, 0)$

(b) What is the force on $q$ at $(0, 0, a)$?

(c) How much work does it take to move $q$ from $(a, 0, 0)$ to $(0, 0, a)$?

A dipole p is situated at the origin , pointing in the z direction.

(a) What is the force on a point charge $q$ at $(a, 0, 0)$

(b) What is the force on $q$ at $(0, 0, a)$?

(c) How much work does it take to move $q$ from $(a, 0, 0)$ to $(0, 0, a)$?

(a) $(a, 0, 0)$ ⇒ $r = a,\ \theta = \frac{\pi}{2},\ \phi = 0,$

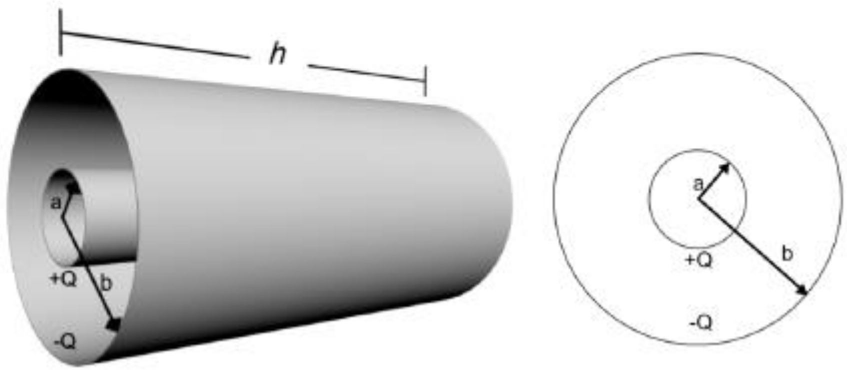
$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 a^3}\hat{\boldsymbol{\theta}} = \frac{p}{4\pi\epsilon_0 a^3}(-\hat{\mathbf{z}}) \qquad \mathbf{F} = q\mathbf{E} = -\frac{pq}{4\pi\epsilon_0 a^3}\hat{\mathbf{z}}.$$

(b) $(0, 0, a)$ ⇒ $r = a,\ \theta = 0,$

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 a^3}(2\hat{\mathbf{r}}) = \frac{2p}{4\pi\epsilon_0 a^3}\hat{\mathbf{z}} \qquad \mathbf{F} = \frac{2pq}{4\pi\epsilon_0 a^3}\hat{\mathbf{z}}.$$

(c)
$$W = q\left[V(0,0,a) - V(a,0,0)\right] = \frac{qp}{4\pi\epsilon_0 a^2}\left[\cos(0) - \cos\left(\frac{\pi}{2}\right)\right] = \frac{pq}{4\pi\epsilon_0 a^2}$$

Consider two nested cylindrical conductors of height $h$ and inner and outer radii $a$ & $b$, respectively, as shown in the figure. A charge $+Q$ is evenly distributed on the outer surface of the inner cylinder, and $-Q$ is uniformly distributed on the inner surface of the outer cylinder. Assume that $h \gg b$ so that the cylinders are effectively infinitely long and end effects can be ignored.




Calculate the magnitude of the electric field between the two cylinders ($a < r < b$).

Calculate the potential difference between the inner and outer cylinders.

Calculate the capacitance of this system.

Calculate the magnitude of the electric field between the two cylinders ($a < r < b$).

For this we use Gauss's Law, with a Gaussian cylinder of radius $r$, height $l$:

$$\oiint_{\text{Gaussian cylinder}} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 2\pi r l E = \frac{Q_{inside}}{\varepsilon_0} = \frac{1}{\varepsilon_0} Q \frac{l}{h}$$

$$E(r)_{a<r<b} = \frac{Q}{2\pi r \varepsilon_0 h}$$

Calculate the potential difference between the inner and outer cylinders.

$$\Delta V = V(a) - V(b) = -\int_b^a \frac{Q}{2\pi r' \varepsilon_0 h} dr'$$

$$= -\frac{Q}{2\pi \varepsilon_0 h} \ln r' |_b^a = \frac{Q}{2\pi \varepsilon_0 h} \ln \frac{b}{a}$$

Calculate the capacitance of this system.

$$C = \frac{|Q|}{|\Delta V|} = \frac{|Q|}{\dfrac{|Q|}{2\pi\varepsilon_0 h}\ln\dfrac{b}{a}} = \frac{2\pi\varepsilon_0 h}{\ln\dfrac{b}{a}}$$

Note that $Q$ cancels out as expected. The capacitance depends only on geometry and $\varepsilon_0$.

Find the electric field energy density (energy per unit volume) as a function of radius at any point between the conducting cylinders. Use that density to determine how much energy resides in a cylindrical shell between the conductors of radius $r$ (with $a < r < b$), height $h$, thickness $dr$, and volume $2\pi rhdr$? Integrate your expression to find the total energy stored in the capacitor.

The electric field energy density stored in the capacitor is

$$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0\left(\frac{Q}{2\pi r\varepsilon_0 h}\right)^2$$

The energy stored in a cylindrical shell of length $l$, radius $r$, and thickness $dr$ is

$$dU = u_E dV = \frac{1}{2}\varepsilon_0 \left(\frac{Q}{2\pi r\varepsilon_0 h}\right)^2 2\pi r h dr = \frac{Q^2}{4\pi\varepsilon_0 h}\frac{dr}{r}$$

Integrating we find that the total energy stored is

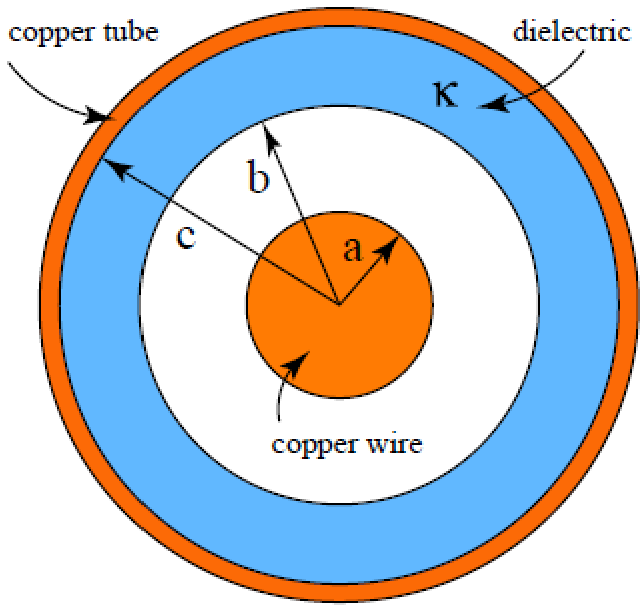
$$U = \int_a^b dU = \int_a^b \frac{Q^2}{4\pi\varepsilon_0 h}\frac{dr}{r} = \frac{Q^2}{4\pi\varepsilon_0 h}\ln\frac{b}{a}$$

$C = 2\pi\varepsilon_0 h/\ln(b/a)$, therefore

$$U = \frac{Q^2}{2C} = \frac{1}{2}C\Delta V^2 = \frac{1}{2}Q\Delta V$$

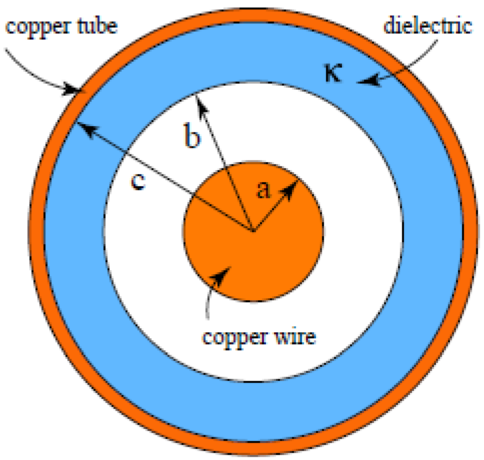
# Coaxial Cable with Dielectric

A certain coaxial cable consists of a copper wire, radius $a$, surrounded by a concentric copper tube of inner radius $c$. The space between is partially filled (from $b$ out to $c$) with material of dielectric constant $\kappa$. The goal of this problem is to find the capacitance per unit length of this cable. You may neglect edge effects.




Assume that the copper wire has uniform positive charge per unit length $\lambda$ and the copper tube has uniform negative charge per unit length on it's inner surface $-\lambda$. Calculate the radial component of the electric field for $0 < r < a$, $a < r < b$, $b < r < c$ and $r > c$ r

Let's apply Gauss's Law $\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{free}}{\kappa\varepsilon_0}$ for each region.




For $a < r < b$: The dielectric constant is $\kappa = 1$

$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{free}}{\varepsilon_0}$$

Gaussian cylinder of radius $r$ such that $a < r < b$, and length $d$,

$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 2\pi Erd \qquad q_{free} = \lambda d$$

$$2\pi Erd = \frac{\lambda d}{\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r},\ a < r < b$$

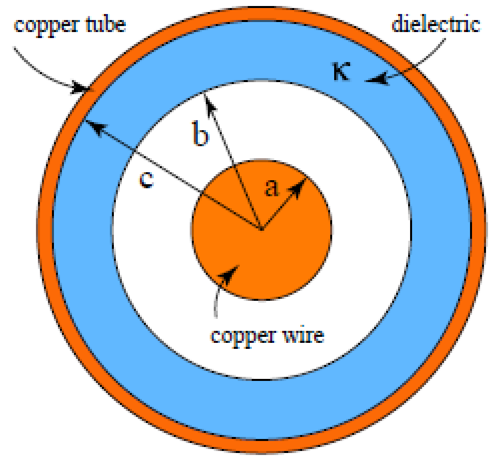
$b < r < c$ The dielectric constant is $\kappa = 1$

Gaussian cylinder of radius $r$ such that $a < r < b$, and length $d$,

Gauss' law becomes

$$2\pi Erd = \frac{\lambda d}{\kappa\varepsilon_0} \Rightarrow E = \frac{\lambda}{2\pi\kappa\varepsilon_0 r},\ b < r < c$$

The region $r < a$ is inside the conductor and for $r > c$ there is no charge enclosed in the Gaussian surface, and so the electric field is zero in those areas.



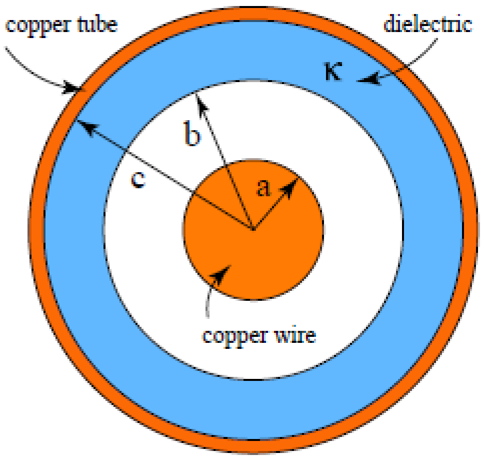

$$\vec{\mathbf{E}}(r) = \begin{cases} \vec{\mathbf{0}}, r < a \\ \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\,\hat{\mathbf{r}},\ a < r < b \\ \frac{\lambda}{2\pi\kappa\varepsilon_0}\frac{1}{r}\,\hat{\mathbf{r}},\ b < r < c \\ \vec{\mathbf{0}},\ r > c\,. \end{cases}$$

What is potential difference between the surfaces $r = b$ and $r = a$?

What about between surfaces $r = c$ and $r = b$ ?

What is the value of $|V(c) - V(a)|$?

The electric potential difference between surfaces $r = a$ and $r = b$ may be found as an integral of electric field along radial direction (independent of the integration path).




$$V(b) - V(a) = -\int_a^b \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\,dr$$

$$= -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{b}{a} = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{a}{b}$$

Between the surfaces $r = c$ and $r = b$,

$$V(c) - V(b) = -\int_b^c \frac{\lambda}{2\pi\kappa\varepsilon_0}\frac{1}{r}\,dr$$

$$= -\frac{\lambda}{2\pi\kappa\varepsilon_0}\ln\frac{c}{b} = \frac{\lambda}{2\pi\kappa\varepsilon_0}\ln\frac{b}{c}$$

Thus the electric potential difference between the wire and the tube is

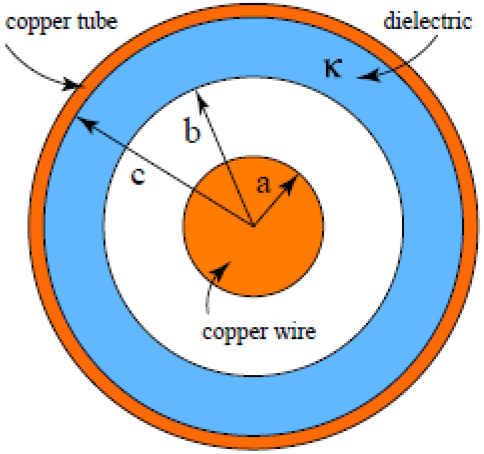
$$V(c) - V(a) = -\int_a^b \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{r}\,dr - \int_b^c \frac{\lambda}{\kappa 2\pi\varepsilon_0}\frac{1}{r}dr$$
$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\ln\frac{a}{b} + \frac{1}{\kappa}\ln\frac{b}{c}\right].$$

Because we are moving from a positive charge to a negative charge, we get a negative voltage difference (recall that $\ln x < 0$ if $x < 1$), so the absolute value of the voltage difference is given by

$$|\Delta V| = \frac{\lambda}{2\pi\varepsilon_0}\left[\ln\frac{b}{a} + \frac{1}{\kappa}\ln\frac{c}{b}\right]$$

What is the capacitance per unit length of the cylindrical arrangement?

The capacitance per unit length is




$$\frac{C}{L} = \frac{Q/L}{|\Delta V|}$$

$$= \frac{\lambda}{\frac{\lambda}{2\pi\varepsilon_0}\left[\ln\frac{b}{a} + \frac{1}{\kappa}\ln\frac{c}{b}\right]}$$

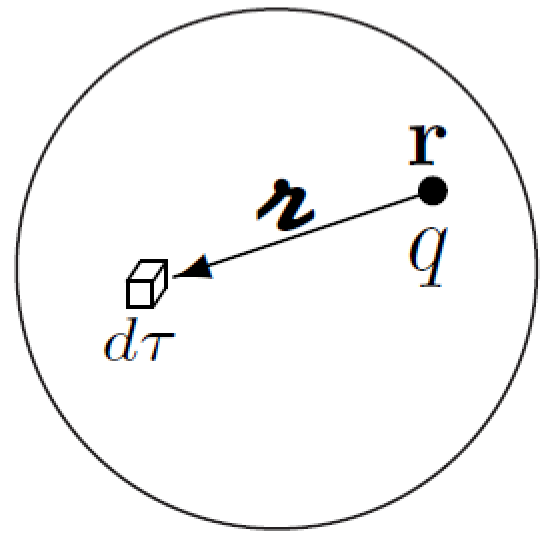
$$= \frac{2\pi\varepsilon_0}{\left[\ln\frac{b}{a} + \frac{1}{\kappa}\ln\frac{c}{b}\right]}$$

Capacitance depends only on the geometry and not on the charge.

# Average Field inside a sphere

The average field due to a point charge $q$ at $\mathbf{r}$ is

$$\mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi R^3\right)} \int \mathbf{E}\, d\tau$$




$$\mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi R^3\right)} \frac{1}{4\pi\epsilon_0} \int q \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^{\,2}}\, d\tau$$

Here $\mathbf{r}$ is the source point, $d\tau$ is the field point, so $\boldsymbol{\mathscr{r}}$ goes from $\mathbf{r}$ to $d\tau$.

$$\mathbf{E}_{\rho} = \frac{1}{4\pi\epsilon_0} \int \rho \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^{\,2}}\, d\tau$$

The field at $\mathbf{r}$ due to uniform charge $\rho$ over the sphere is $\mathbf{E}_{\rho}$

This time $d\tau$ is the source point and $\mathbf{r}$ is the field point, so $\boldsymbol{\mathscr{r}}$ goes from $d\tau$ to $\mathbf{r}$, $\quad \rho = -q/\left(\frac{4}{3}\pi R^3\right)$

$$\mathbf{E}_{\text{ave}} = \mathbf{E}_\rho \quad \Longleftrightarrow \quad \rho = -q/\left(\tfrac{4}{3}\pi R^3\right)$$

$$\mathbf{E}_\rho = \frac{1}{3\epsilon_0}\rho\mathbf{r}$$

$$= -\frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{R^3}$$

$$= -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3}$$

# Average Field inside an arbitrary charge distribution

superposition principle to generalize to an arbitrary charge distribution.

If there are many charges inside the sphere,

$\mathbf{E}_{\text{ave}}$ is the sum of the individual averages,

$\mathbf{p}_{\text{tot}}$ → sum of the individual dipole moments

$$\mathbf{E}_{\text{ave}} = -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3}$$

# Average Field of a dipole over a spherical volume of radius R

$$\boxed{\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).}$$

$$\begin{aligned}
\mathbf{E}_{\text{dip}} &= \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) \\
&= \frac{p}{4\pi\epsilon_0 r^3}\,[2\cos\theta(\sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}) \\
&\quad + \sin\theta(\cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}})] \\
&= \frac{p}{4\pi\epsilon_0 r^3}\left[3\sin\theta\cos\theta\cos\phi\,\hat{\mathbf{x}} + 3\sin\theta\cos\theta\sin\phi\,\hat{\mathbf{y}} + \underbrace{\left(2\cos^2\theta - \sin^2\theta\right)}_{=3\cos^2\theta-1}\,\hat{\mathbf{z}}\right]
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}_{\text{ave}} &= \frac{1}{\left(\frac{4}{3}\pi R^3\right)}\int \mathbf{E}_{\text{dip}}\,d\tau \\
&= \frac{1}{\left(\frac{4}{3}\pi R^3\right)}\left(\frac{p}{4\pi\epsilon_0}\right)\int \frac{1}{r^3}\,[3\sin\theta\cos\theta(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) + \left(3\cos^2\theta - 1\right)\,\hat{\mathbf{z}}]\,r^2\sin\theta\,dr\,d\theta\,d\phi
\end{aligned}$$

$$\mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi R^3\right)} \int \mathbf{E}_{\text{dip}}\, d\tau$$

$$= \frac{1}{\left(\frac{4}{3}\pi R^3\right)} \left(\frac{p}{4\pi\epsilon_0}\right) \int \frac{1}{r^3} \left[3\sin\theta\cos\theta(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}) + \left(3\cos^2\theta - 1\right)\,\hat{\mathbf{z}}\right] r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$\int_0^{2\pi} \cos\phi\, d\phi = \int_0^{2\pi} \sin\phi\, d\phi = 0, \quad \Longrightarrow \quad \hat{\mathbf{x}} \text{ and } \hat{\mathbf{y}} \text{ terms drop out}$$

**0**

$$\mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi R^3\right)} \left(\frac{p}{4\pi\epsilon_0}\right) 2\pi \int_0^R \frac{1}{r}\, dr \int_0^{\pi} \left(3\cos^2\theta - 1\right) \sin\theta\, d\theta$$

$$\mathbf{E}_{\text{ave}} = \mathbf{0}$$

$$\mathbf{E} = \mathbf{A}\delta^3(\mathbf{r})$$ ⇒ $\mathbf{E}$ within the $\epsilon$-sphere to be a delta function

$$\mathbf{E}_{\text{ave}} = \frac{1}{\left(\frac{4}{3}\pi R^3\right)} \int \mathbf{A}\delta^3(\mathbf{r})\, d\tau$$

$$= \frac{\mathbf{A}}{\left(\frac{4}{3}\pi R^3\right)}$$

$$= -\frac{\mathbf{p}}{4\pi\epsilon_0 R^3}$$

$$\mathbf{A} = -\frac{\mathbf{p}}{3\epsilon_0}$$

$$\mathbf{E} = -\frac{\mathbf{p}}{3\epsilon_0}\delta^3(\mathbf{r})$$

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right] - \frac{1}{3\epsilon_0}\mathbf{p}\,\delta^3(\mathbf{r})$$ ⇒ *true* field of a dipole