Mathematical Physics - 3

4th Semester BSc Physics(Honours)

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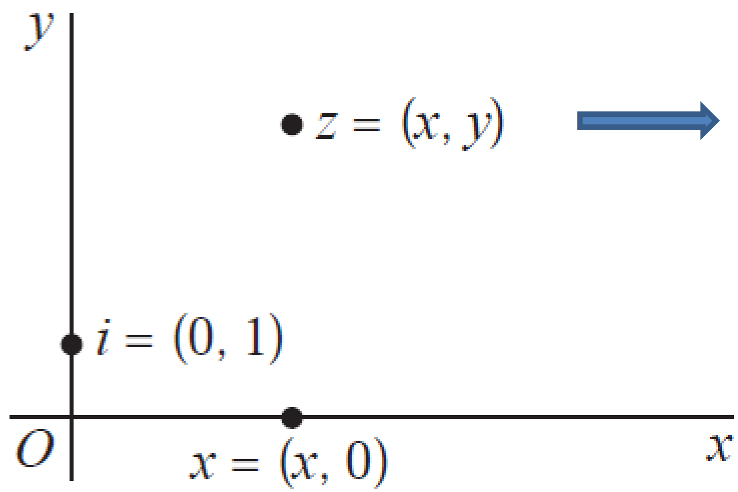


$$z = x + iy$$

$$|z| = \sqrt{x^2 + y^2}$$

$$i^2 = -1$$

It is customary to denote a complex number (x, y) by z

$$z = (x, y)$$

$$x = \operatorname{Re} z, \; y = \operatorname{Im} z$$

The *sum* $z_1 + z_2$ and *product* $z_1 z_2$ of two complex numbers

$$z_1 = (x_1, y_1) \quad \text{and} \quad z_2 = (x_2, y_2)$$

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2)$$

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2).$$

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0),$$

$$(x_1, 0)(x_2, 0) = (x_1 x_2, 0).$$

$$z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1 \qquad \text{commutative laws}$$

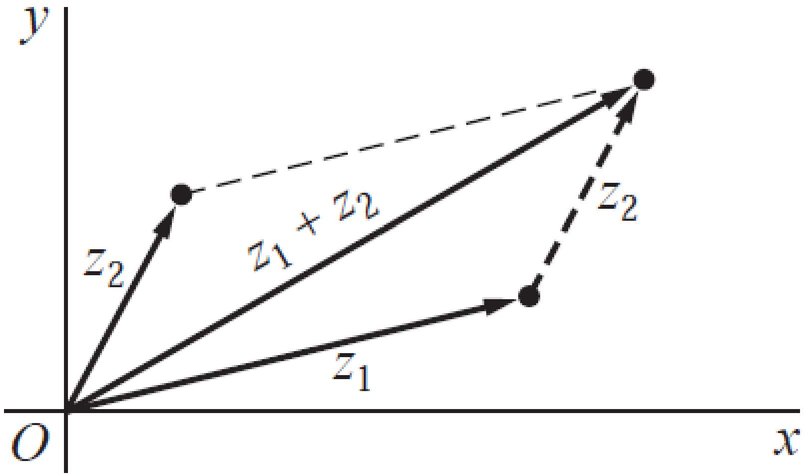
$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \quad (z_1 z_2) z_3 = z_1 (z_2 z_3)$$

associative laws

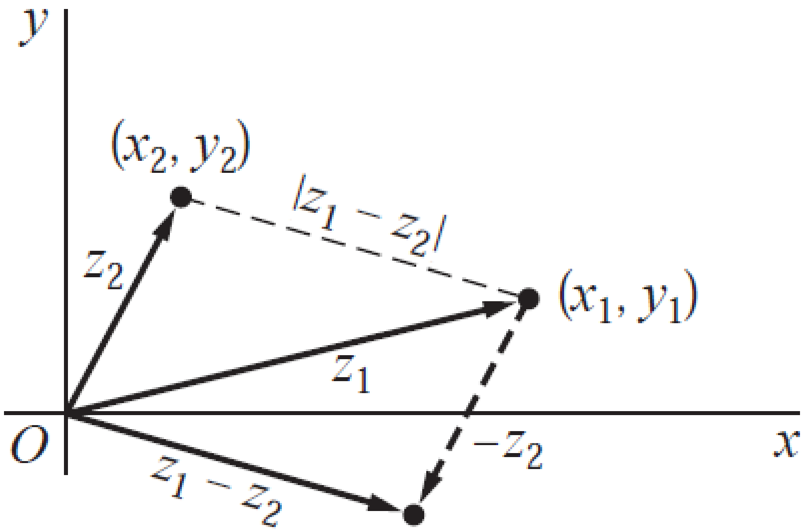
$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) \qquad (z \neq 0)$$

The inverse z^{-1} is not defined when $z = 0$.

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}$$



$$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$$



$$z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$$

$$z = x + iy$$

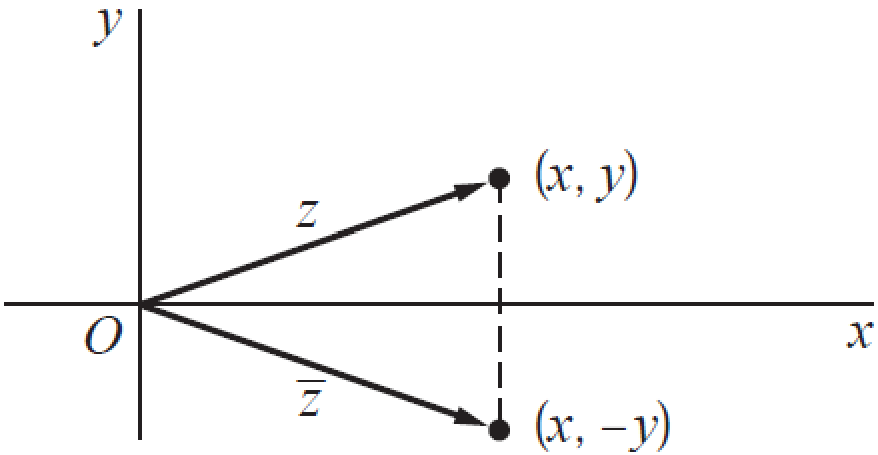
$$\operatorname{Re} z = x, \quad \operatorname{Im} z = y$$

$$|z| = \sqrt{x^2 + y^2}$$

$$|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2$$

$$\operatorname{Re} z \leq |\operatorname{Re} z| \leq |z| \quad \text{and} \quad \operatorname{Im} z \leq |\operatorname{Im} z| \leq |z|.$$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$ $\Rightarrow$ *triangle inequality.*



$$z = x + iy$$

$$\bar{z} = x - iy \quad \textit{complex conjugate}$$

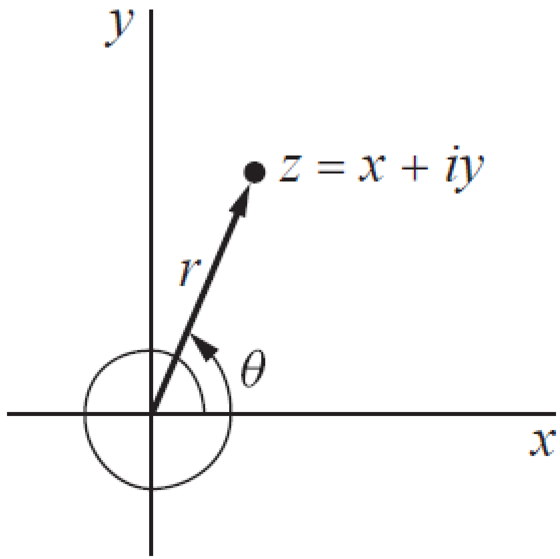
$$\bar{\bar{z}} = z \quad \text{and} \quad |\bar{z}| = |z|$$

$$\overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2).$$

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2} \qquad \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}} \qquad (z_2 \neq 0)$$

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$$



$$z = x + iy$$

$$x = r\cos\theta \quad y = r\sin\theta$$

$$z = r(\cos\theta + i\sin\theta)$$

$$r = |z|$$

As in calculus, θ has an infinite number of possible values, including negative ones, that differ by integral multiples of 2π. Each value of θ is called an *argument* of z, and the set of all such values is denoted by $\arg z$.

The *principal value* of $\arg z$, denoted by $\operatorname{Arg} z$,

is that unique value Θ such that $-\pi < \Theta \leq \pi$.

$$\arg z = \operatorname{Arg} z + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Also, when z is a negative real number, $\operatorname{Arg} z$ has value π, not $-\pi$.

$$\begin{aligned} e^{i\theta_1} e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = e^{i(\theta_1 + \theta_2)}. \end{aligned}$$

- $\operatorname{Arg} i = \frac{\pi}{2}$,
- $\operatorname{Arg} 1 = 0$,
- $\operatorname{Arg}(-1) = \pi$,
- $\operatorname{Arg}(1 - i) = -\frac{\pi}{4}$,
- $\operatorname{Arg}(-i) = -\frac{\pi}{2}, \ldots$

- Convenient notation: $e^{i\theta} = \cos\theta + i\sin\theta$.

- Note: $e^{i(\theta+2\pi)} = e^{i\theta} = e^{i(\theta+4\pi)} = \cdots = e^{i(\theta+2k\pi)}$

- $e^{i\frac{\pi}{2}} = i$,
- $e^{i\pi} = -1$,
- $e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$,
- $e^{2\pi i} = 1, \ldots$

$$z_1 = r_1 e^{i\theta_1} \qquad z_2 = r_2 e^{i\theta_2}$$

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i\theta_1} e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} \cdot \frac{e^{i\theta_1} e^{-i\theta_2}}{e^{i\theta_2} e^{-i\theta_2}} = \frac{r_1}{r_2} \cdot \frac{e^{i(\theta_1 - \theta_2)}}{e^{i0}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

$$z^n = r^n e^{in\theta} \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

$$(e^{i\theta})^n = e^{in\theta} \quad (n = 0, \pm 1, \pm 2, \ldots)$$

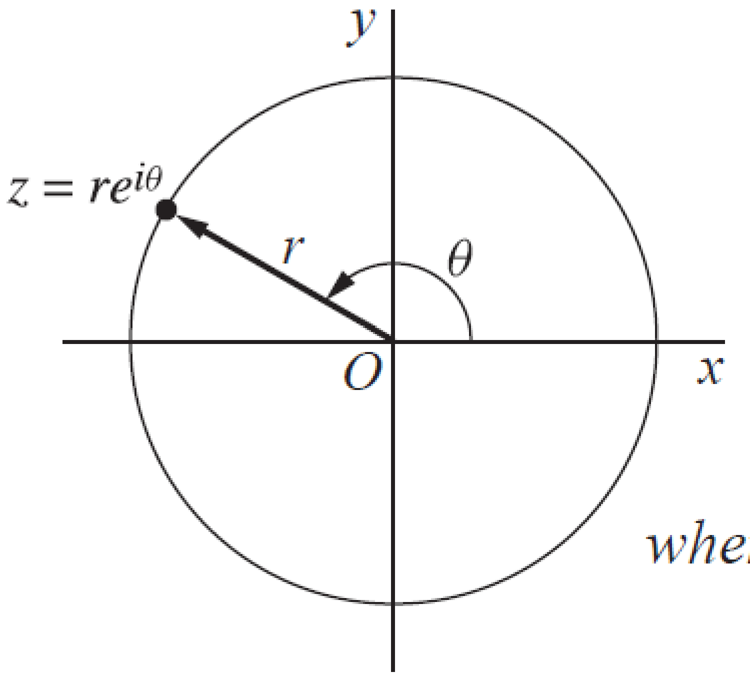
de Moivre's formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad (n = 0, \pm 1, \pm 2, \ldots)$$

Use de Moivre's formula to derive the following trigonometric identities:

(a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$; *(b)* $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

Roots of Complex number



two nonzero complex numbers

$$z_1 = r_1 e^{i\theta_1} \quad \textit{and} \quad z_2 = r_2 e^{i\theta_2}$$

are equal if and only if

$$r_1 = r_2 \quad \textit{and} \quad \theta_1 = \theta_2 + 2k\pi$$

where k is some integer $(k = 0, \pm 1, \pm 2, \ldots)$

The method starts with the fact that an nth root of z_0

is a nonzero number $z = re^{i\theta}$ such that $z^n = z_0$,

$$r^n e^{in\theta} = r_0 e^{i\theta_0}$$

$$r^n e^{in\theta} = r_0 e^{i\theta_0}$$

$$r^n = r_0 \quad \text{and} \quad n\theta = \theta_0 + 2k\pi$$

$$(k = 0, \pm 1, \pm 2, \ldots)$$

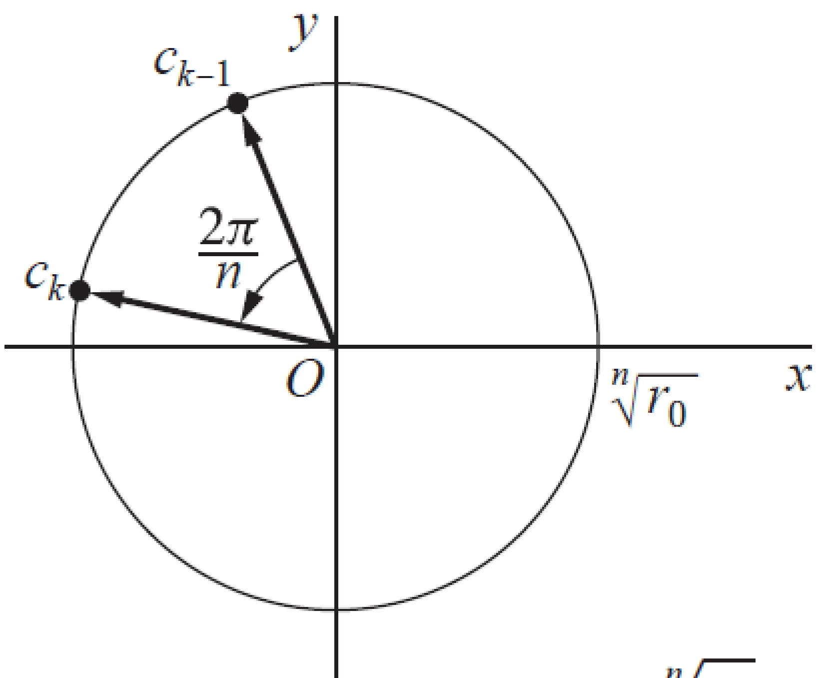
$$r = \sqrt[n]{r_0}$$

$$\theta = \frac{\theta_0 + 2k\pi}{n} = \frac{\theta_0}{n} + \frac{2k\pi}{n}$$

$$z = \sqrt[n]{r_0} \exp\left[i \left(\frac{\theta_0}{n} + \frac{2k\pi}{n} \right) \right] \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

↑

nth roots of z_0



$$c_k = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right]$$

$$(k = 0, 1, 2, \ldots, n-1)$$

$$c_k = \sqrt[n]{r_0} \exp\left(i\frac{\theta_0}{n}\right) \exp\left(i\frac{2k\pi}{n}\right)$$

c_0 is referred to as the *principal root.*

$$\omega_n = \exp\left(i\frac{2\pi}{n}\right) \Rightarrow \omega_n^k = \exp\left(i\frac{2k\pi}{n}\right)$$

$$c_k = c_0\omega_n^k \quad (k = 0, 1, 2, \ldots, n-1)$$

ω_n represents a counterclockwise rotation through $2\pi/n$ radians.

Square roots of $4i$

$$4i = 4e^{i\frac{\pi}{2}}, \quad \text{so} \quad \rho = 4, \varphi = \frac{\pi}{2} \text{ and } n = 2.$$

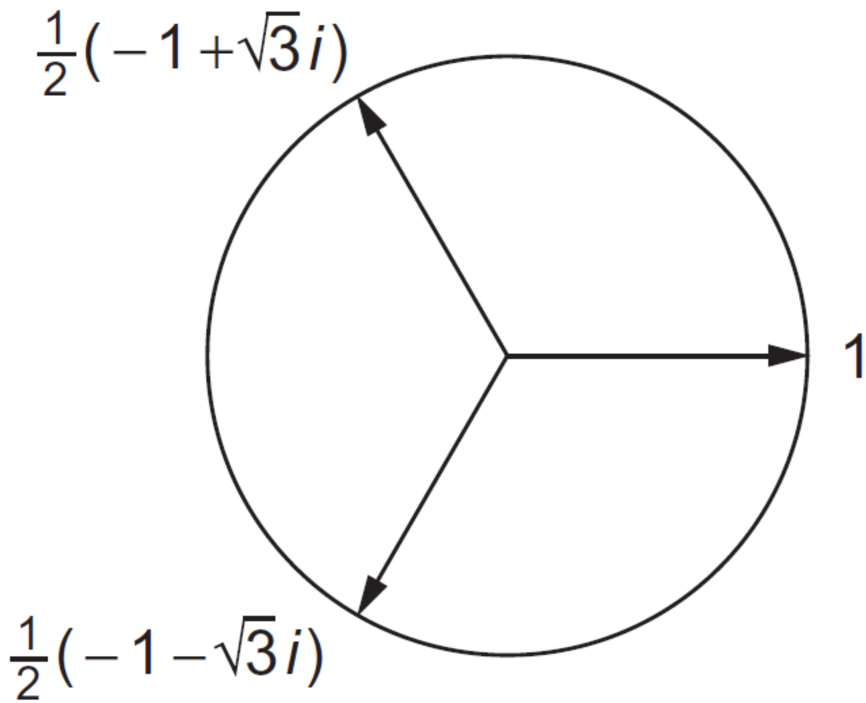
$$\begin{aligned}
(4i)^{\frac{1}{2}} &= \sqrt{4} \cdot e^{i\left(\frac{\pi}{4} + \frac{2k\pi}{2}\right)}, k = 0, 1 \\
&= \begin{cases} 2 \cdot e^{i\frac{\pi}{4}} & \text{if } k = 0 \\ 2 \cdot e^{i\left(\frac{\pi}{4} + \pi\right)} & \text{if } k = 1 \end{cases} \\
&= \pm(\sqrt{2} + i\sqrt{2}).
\end{aligned}$$

Cubed roots of -8

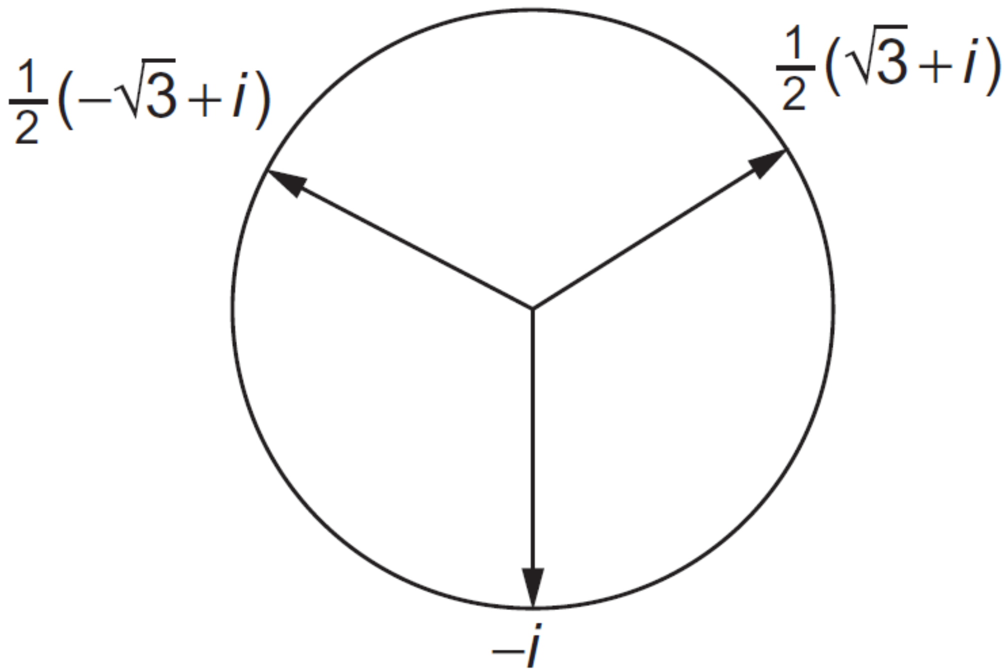
$$-8 = 8e^{i\pi}, \quad \text{so} \quad \rho = 8, \varphi = \pi \text{ and } n = 3.$$

$$\begin{aligned} (-8)^{\frac{1}{3}} &= \sqrt[3]{8} \cdot e^{i\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)}, k = 0, 1, 2 \\ &= \begin{cases} 2 \cdot e^{i\frac{\pi}{3}} & \text{if } k = 0 \\ 2 \cdot e^{i\pi} = -2 & \text{if } k = 1 \\ 2 \cdot e^{i\frac{5\pi}{3}} & \text{if } k = 2. \end{cases} \end{aligned}$$

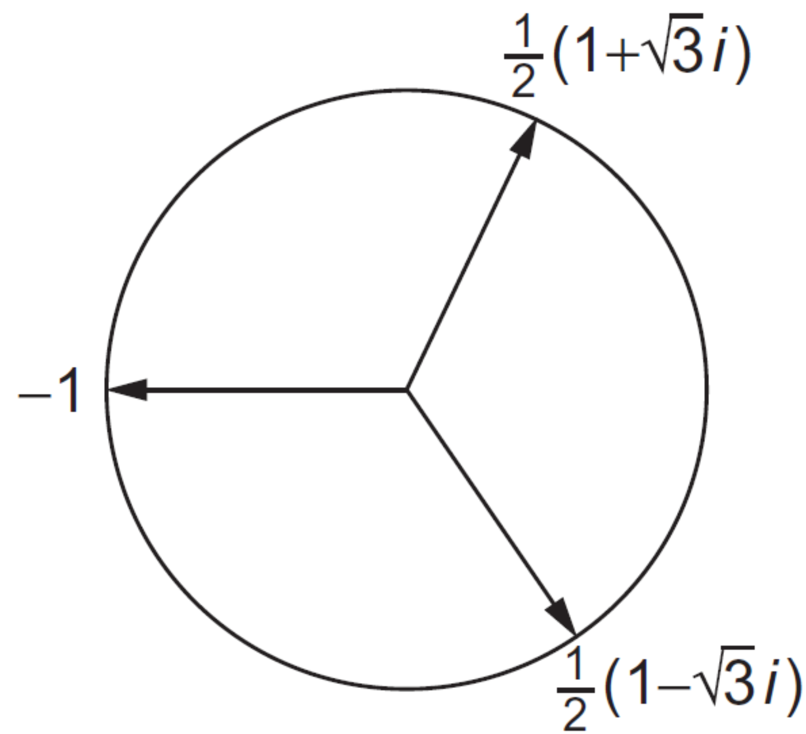
Cube roots: $1^{1/3}$



Cube roots: $i^{1/3}$



Cube roots: $(-1)^{1/3}$



Show that

(a) $\cos n\theta = \cos^n \theta - \binom{n}{2} \cos^{n-2} \theta \sin^2 \theta + \binom{n}{4} \cos^{n-4} \theta \sin^4 \theta - \cdots,$

(b) $\sin n\theta = \binom{n}{1} \cos^{n-1} \theta \sin \theta - \binom{n}{3} \cos^{n-3} \theta \sin^3 \theta + \cdots.$

$$e^{in\theta} = \cos n\theta + i \sin n\theta = (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = \sum_{\nu=0}^{n} \binom{n}{\nu} \cos^{n-\nu}\theta (i\sin\theta)^\nu.$$

Separating real and imaginary parts we have

$$\cos n\theta = \sum_{\nu=0}^{[n/2]} (-1)^\nu \binom{n}{2\nu} \cos^{n-2\nu}\theta \sin^{2\nu}\theta,$$

$$\sin n\theta = \sum_{\nu=0}^{[n/2]} (-1)^\nu \binom{n}{2\nu+1} \cos^{n-2\nu-1}\theta \sin^{2\nu+1}\theta.$$

Prove that

(a) $$\sum_{n=0}^{N-1} \cos nx = \frac{\sin(Nx/2)}{\sin x/2} \cos(N-1)\frac{x}{2},$$

(b) $$\sum_{n=0}^{N-1} \sin nx = \frac{\sin(Nx/2)}{\sin x/2} \sin(N-1)\frac{x}{2}.$$

These series occur in the analysis of the multiple-slit diffraction pattern.

$$\sum_{n=0}^{N-1} (e^{ix})^n = \frac{1 - e^{iNx}}{1 - e^{ix}} = \frac{e^{iNx/2}}{e^{ix/2}} \frac{e^{iNx/2} - e^{-iNx/2}}{e^{ix/2} - e^{-ix/2}}$$

$$= e^{i(N-1)x/2} \sin(Nx/2)/\sin(x/2).$$

Now take real and imaginary parts to get the result.

Functions of Complex Variable

If we denote the real and imaginary parts of $f(z)$ by u and v,

$$f(z) = u(x, y) + iv(x, y)$$

A function $f(z)$ that is single-valued in some domain R is *differentiable* at the point z in R if the *derivative*

$$f'(z) = \lim_{\Delta z \to 0} \left[\frac{f(z + \Delta z) - f(z)}{\Delta z} \right] \tag{24.1}$$

exists and is unique, in that its value does not depend upon the direction in the Argand diagram from which Δz tends to zero.

The Cauchy-Riemann Relations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are the famous **Cauchy-Riemann** conditions. They were discovered by Cauchy and used extensively by Riemann in his development of complex variable theory. These Cauchy-Riemann conditions are necessary for the existence of a derivative of $f(z)$. That is, in order for df/dz to exist, the Cauchy-Riemann conditions must hold.

Derivatives of Analytic Function

$$f'(z) = \frac{\partial f}{\partial x} \quad \Rightarrow \quad \textbf{if } f(z) \textbf{ is analytic}$$

$$\delta f = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

Cauchy-Riemann equations

$$\delta f = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\delta x + i\delta y). \quad \rightarrow \delta z$$

$$\frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \Bigg\} \quad \frac{\partial f}{\partial x}$$

Find the analytic function

$$w(z) = u(x, y) + i v(x, y)$$

(a) if $u(x, y) = x^3 - 3xy^2$,

(b) if $v(x, y) = e^{-y} \sin x$

Two-dimensional irrotational fluid flow is conveniently described by a complex potential $f(z) = u(x, v) + iv(x, y)$. We label the real part, $u(x, y)$, the velocity potential, and the imaginary part, $v(x, y)$, the stream function. The fluid velocity $\mathbf{V}$ is given by $\mathbf{V} = \nabla u$. If $f(z)$ is analytic:

(a) Show that $df/dz = V_x - iV_y$.

(b) Show that $\nabla \cdot \mathbf{V} = 0$ (no sources or sinks).

(c) Show that $\nabla \times \mathbf{V} = 0$ (irrotational, nonturbulent flow).