2.4 Different kinds of wave

In the discussions above we have assumed that y is displacement of the particles of the medium from their mean positions. This is true for mechanical wave like that in a stretched string. But there are different kinds of wave and accordingly y may represent different physical quantities. For example, to describe sound waves the more convenient physical quantity is *excess pressure* over normal pressure. In the region of compression pressure is slightly above normal and in the region of rarefaction pressure is slightly below normal pressure. Hence for sound wave y stands for *excess pressure.* For electromagnetic waves what really propagate are varying electric field and associated magnetic field. And for electromagnetic waves there is no need for a medium. Hence for electromagnetic wave y stands for *any component* of these fields.

Hence for the sake of generalisation and convenience we may choose the symbol ψ for y and we call it *wave function*. Instead of giving a particular name like displacement, electric field, ext., we call $\psi(x,t)$ as the *disturbance* at position x and time t.

Thus harmonic wave proceeding along positive X-axis is represented by the wave function :

$$\psi(x,t) = a \cos k(ct - x)$$

Any arbitrary wave proceeding along positive X-axis is represented by the wave function :

$$\psi(x,t) = f(ct - x)$$

Differential equation for a wave proceeding along X-axis is

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}.$$

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2.5 Propagation of a wave through three media

(1) *Transverse wave through a stretched string* :

We consider a thin perfectly flexible string stretched under tension T and its length coinciding with the X-axis. If we pluck or strike the string at a point momentarily, a transverse wave would proceed along its length. Instantaneous position of a very small portion AB of length Δl of the string is shown in Fig.2.8. Let us consider the dynamics of the situation and get the differential equation of the wave.

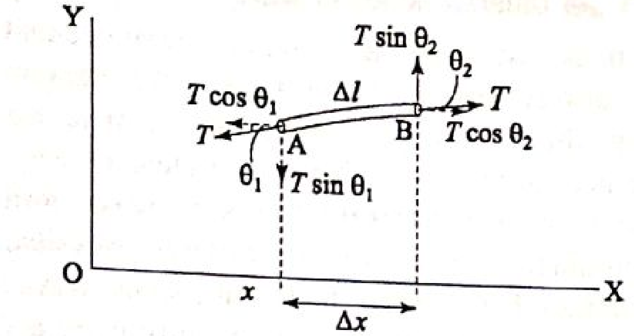

Fig. 2.8

As the string is perfectly flexible, tension will be the *same* throughout the string and act *tangentially* at every point on AB. As displacement y of A is very small, the angles θ_1 and θ_2 are very small. The two components of T along X-axis at A and B ($T\cos\theta_1$ and $T\cos\theta_2$) are equal and opposite, as there is no net force along X-axis, vibrations are along Y-axis.

The net force along Y-axis is

$$T \sin\theta_2 - T\sin\theta_1 = T\,[\tan\theta_2 - \tan\theta_1]$$

$$= T\left[\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right]$$

$$= T\left[\left(\frac{\partial y}{\partial x}\right)_A + \frac{\partial}{\partial x}\left(\left(\frac{\partial y}{\partial x}\right)\Delta x - \left(\frac{\partial y}{\partial x}\right)_A\right)\right]$$

$$= T\left(\frac{\partial^2 y}{\partial x^2}\right)\Delta x$$

Here we have used the fact that when θ is very small,

$$\sin\theta = \tan\theta = \text{slope} = \left(\frac{\partial y}{\partial x}\right)$$

If *mass per unit length* of the string is m, mass of the portion AB is $m\Delta l \approx m\Delta x$, as Δl is very small.

Acceleration of the small portion is $\frac{\partial^2 y}{\partial t^2}$. By Newton's law :

$$T\left(\frac{\partial^2 y}{\partial x^2}\right)\Delta x = m\Delta x\,\frac{\partial^2 y}{\partial t^2}$$

$$\text{or,}\quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left(\frac{\partial^2 y}{\partial x^2}\right)$$

This is the differential equation of transverse wave through a stretched string. Comparing this with the eqn. 2.3, we see that the speed of the *transverse* wave in a *stretched string* is given by

$$c = \sqrt{\frac{T}{m}} \quad \text{..................2.4}$$

We see the velocity depends on *properties* of the medium.

(2) *Longitudinal wave through a solid medium* :

We consider a thin bar situated along X-axis through which a longitudinal wave is passing. Different layers of the bar are vibrating with different phases along X-axis. We consider two planes A and B at distances x and $x + \delta x$ from the origin O, Fig. 2.9. Suppose at a particular instant t the two layers are at A′ and B′, their displacements from mean positions are y and $y + dy$ respectively.

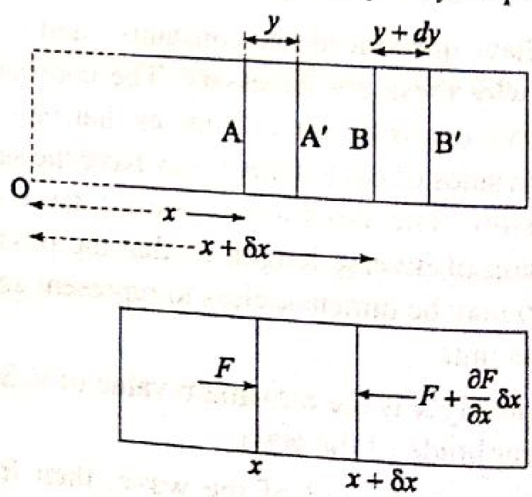

Fig. 2.9

Displacement y of a layer at a particular instant during vibration is a function of x. Rate of change of y with x is $\frac{\partial y}{\partial x}$. $\therefore\ dy = \frac{\partial y}{\partial x}\delta x$.

The original distance between the two planes was δx. The new distance at the instant t is $\delta x + dy$.

$\therefore$ Change in the distance is $dy = \frac{\partial y}{\partial x}\delta x.$

Longitudinal strain produced in that portion of the rod is

$$\frac{dy}{\delta x}=\frac{\partial y}{\partial x} \quad ....2.5$$

Forces developed in the two sides of the layer between A and B are respectively F and $F+\frac{\partial F}{\partial x}\delta x$, as shown in the figure. The two opposite forces F and F produce the compressive strain $\left(-\frac{\partial y}{\partial x}\right)$ in the layer and the unbalanced force $-\frac{\partial F}{\partial x}\delta x$ produce the acceleration.

Let S be the area of cross-section of the layer. As S is assumed to be small, force is same throughout the cross-section.

By Hooke's law: Young's modulus,

$$Y=\frac{\text{stress}}{\text{strain}}=-\frac{F/S}{\partial y/\partial x}$$

$$\therefore\ F=-YS\frac{\partial y}{\partial x} \quad ....(i)$$

If ρ is the density of solid, mass of the thin portion is $\rho S\delta x$.

Acceleration of the layer AB is $\frac{\partial^2 y}{\partial t^2}$

By Newton's law : $-\frac{\partial F}{\partial x}\delta x=\rho S\delta x\frac{\partial^2 y}{\partial t^2}$

Substituting for F from eqn.(i) we get

$$YS\frac{\partial^2 y}{\partial x^2}\delta x=\rho S\delta x\frac{\partial^2 y}{\partial t^2}$$

$$\text{or,}\quad \frac{\partial^2 y}{\partial t^2}=\frac{Y}{\rho}\frac{\partial^2 y}{\partial x^2}$$

This is the differential equation of longitudinal wave through an elastic solid. Comparing this with the eqn. 2.3, we see that the speed of the *longitudinal wave* in an *elastic solid* is given by

$$c=\sqrt{\frac{Y}{\rho}} \quad ....2.6$$

Comparison of transverse and longitudinal wave :

Velocities of transverse and longitudinal waves in solid are given by

$$c_t=\sqrt{\frac{T}{m}} \text{ and } c_l=\sqrt{\frac{Y}{\rho}}$$

We notice that tension T in the string is playing the role elastic modulus Y, providing the restoring force necessary for vibration.

Again, mass per unit length, $m=S\cdot 1\cdot\rho$, S = area of cross-section of the string.

$$c_t=\sqrt{\frac{T}{S\cdot\rho}}$$

For the string we can write, Young's modulus, $Y=\frac{T/S}{l/L}$, where l is increase in length because of the tension T and L is original length.

We find that c_t can be equal to c_l if $Y = T/S$, i.e. $l/L = 1$. But to produce extension (l) equal to original length (L), we have to apply a tension, which far exceeds the elastic limit of the material; the string breaks down. Therefore velocity of longitudinal wave is much *larger* than velocity of transverse wave.

(3) ***Velocity of a longitudinal wave through an elastic fluid :***

We consider a thin cylinder of an elastic fluid (liquid or gas) situated along X-axis through which a longitudinal wave is passing. Different layers of the fluid are vibrating with different phases along X-axis. We consider two planes A and B at distances x and $x+\delta x$ from the origin O respectively, Fig.2.10. Suppose at a particular instant t the two layers are at A′ and B′, their displacements from mean positions are y and $y+dy$ respectively. At a particular instant, displacement y of a layer during vibration is a function of x.

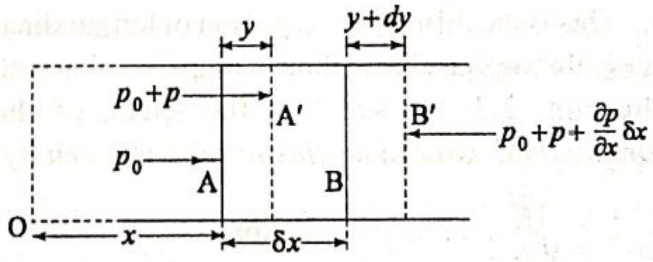

Fig. 2.10

Rate of change of y with x is $\frac{\partial y}{\partial x}$. $\therefore\ dy=\frac{\partial y}{\partial x}\delta x$

If S be the area of cross-section of the cylinder, initial volume of the layer AB of the fluid is $S\delta x$. The initial distance between the two planes was δx. The new distance at the instant t is $\delta x + dy$.

$\therefore$ Change in volume of the layer is $S\delta y$.

$\therefore$ Volume strain produced in the layer is

$$e=\frac{S\delta y}{S\delta x}=\frac{\delta y}{\delta x}.$$

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Pressures developed in the two sides of the layer between A and B are respectively p and $p+\frac{\partial p}{\partial x}\delta x$, as shown in the figure. The equilibrium pressure in the medium was p_0 when the wave is absent. The two opposite pressures p and p constitute the stress producing the strain in the layer and the unbalanced force $S\frac{\delta p}{\delta x}\delta x$ produce the acceleration.

By Hooke's law: Bulk modulus,

$$K=\frac{\text{stress}}{\text{strain}}=-\frac{p}{\frac{\partial y}{\partial x}}$$

$$\therefore p=-K\frac{\partial y}{\partial x}\text{ ....(i)}$$

If ρ is the density of fluid, mass of the thin portion is $\rho S\delta x$.

Acceleration of the layer AB is $\frac{\partial^2 y}{\partial t^2}$

By Newton's law : $S\frac{\partial p}{\partial x}\delta x=-\rho S\delta x\frac{\partial^2 y}{\partial t^2}$

Negative sign indicates that the acceleration is opposite to the X-axis.

Substituting from (i) we get,

$$SK\frac{\partial^2 y}{\partial x^2}\delta x=\rho S\delta x\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial t^2}=\frac{K}{\rho}\frac{\partial^2 y}{\partial x^2}$$

This is the differential equation of longitudinal wave through an elastic fluid. Comparing this with the eqn. 2.3, we see that the speed of the *longitudinal wave* in an *elastic solid* is given by

$$c=\sqrt{\frac{K}{\rho}}\text{ ....2.6a}$$

For a fluid we see the bulk modulus comes into play to produce rarefaction and compression of the successive layers.

For gaseous medium, the compressions and rarefactions take place in adiabatic condition. In adiabatic condition in a perfect gas, we know pV^γ = constant, where $\gamma=C_p/C_v$, C_p and C_v are specific heats of gas at constant pressure and constant volume respectively.

$$\therefore dp\cdot V^\gamma+p\cdot\gamma V^{\gamma-1}dV=0.$$

$$\therefore \gamma p=-\frac{dp\cdot V}{dV}=-\frac{dp}{dV/V}=K\text{ (bulk modulus)}$$

We see adiabatic bulk modulus of gas is γp. Therefore for a gaseous medium velocity of longitudinal wave is

$$c=\sqrt{\frac{\gamma p}{\rho}}\text{ ....2.6c}$$

The pressures (p, p) developed in the layer AB of Fig.2.10 to produce rarefaction (or compression) in the layer are in fact the pressure below (or above) the standard pressure (p_0) in absence of sound wave. As the sound wave moves through air, this excess pressure (p) advances. This pressure (p) is called the acoustic pressure. Hence sound wave may be considered either as displacement wave of the layers or as the pressure wave.

Displacement (y) of a layer at position x and time t is given by

$$y=a\sin(\omega t-kx)$$

$$\text{Acoustic pressure, } p=-K\frac{\partial y}{\partial x}$$

$$=Kka\cos(\omega t-kx)$$

$$=c^2\rho ka\cos(\omega t-kx)$$

$$=p_m\sin\left(\omega t-kx+\frac{\pi}{2}\right)$$

We note that the phase of pressure wave [$p(x,t)$] is $\frac{\pi}{2}$ ahead of the displacement wave [$y(x,t)$]. That means when displacement from equilibrium at point is a maximum or a minimum, the excess pressure there is zero; when the displacement at a point is zero, the excess or deficiency of pressure is a maximum.

The maximum variation of pressure from equilibrium or pressure amplitude is given by

$$p_m=c^2\rho ka=c^2\rho\frac{2\pi}{\lambda}a=2\pi cn\rho a\text{..................2.7}$$

2.6 Plane Wave

We have seen that a harmonic wave proceeding along positive X-axis is represented by the wave function

$$\psi(x,t)=a\cos(\omega t-kx)\text{................2.1}$$

Here we know $\omega=\frac{2\pi}{T}$ and $k=\frac{2\pi}{\lambda}$.

Now we imagine that the above wave is passing through three dimensional medium. The value of ψ does not depend upon y and z coordinates of

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any point in that three dimensional space. Hence the phase ($\omega t - kx$) of the wave has the same value at any instant at all points in any plane imagined parallel to the YZ-plane, Fig.2.11. Thus any plane parallel to the YZ-plane is the locus of point having the same phase of vibration. Such a plane is called wavefront.

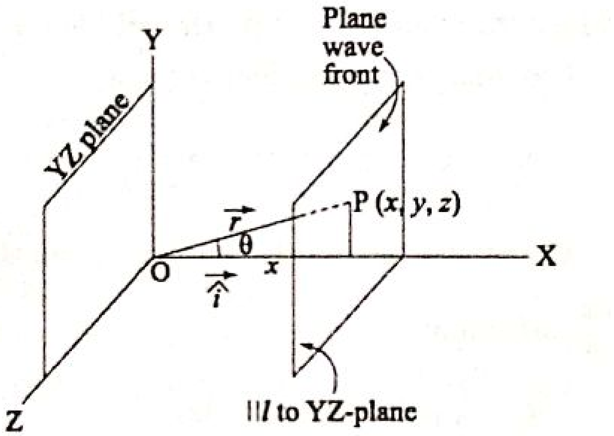

Fig. 2.11

Wavefront : A plane in the medium at every point of which phase of vibration is the same is called the wavefront of the wave.

Therefore the above equation 2.1 represents a wave whose *wavefronts* are *planes parallel to the YZ-planes*. In other words this equation represents a plane wave. A plane wave has successive parallel plane wavefronts. We notice that a plane wave proceed in a particular direction and *does not spread* in the lateral direction.

The above wave is proceeding along X-axis, its direction is the unit vector $\hat{i}$ along X-axis. Now we like to get the equation for a plane wave proceeding along *any arbitrary* direction.

We choose a point P having position vector $\vec{r}(x,y,z)$ on a particular wavefront, as shown in the Fig.2.11. From the figure we can write

$$\omega t - kx = \omega t - kr\cos\theta = \omega t - k\vec{r}.\hat{i}$$

Thus a plane wave proceeding along the direction $\hat{i}$ is represented by

$$\psi(\vec{r},t) = a\cos\left(\omega t - k\vec{r}.\hat{i}\right)$$

Therefore a plane wave proceeding along any arbitrary direction $\hat{k}$ can be written as

$$\psi(\vec{r},t) = a\cos\left(\omega t - k\vec{r}.\hat{k}\right)$$

Now we construct a vector $\vec{k}$ whose magnitude is $k = 2\pi/\lambda$ and whose direction is the direction of propagation $\hat{k}$ of the wave, i.e., vector $\vec{k} = k\hat{k}$.

Then we can write the equation of plane wave proceeding along $\vec{k}$ as

$$\psi(\vec{r},t) = a\cos\left(\omega t - \vec{k}.\vec{r}\right) \quad \text{.................2.8}$$

This vector $\vec{k}$ is called the *propagation vector* of the wave; it is a characteristic of the wave. In Fig.2.12, we can see such a plane wave.

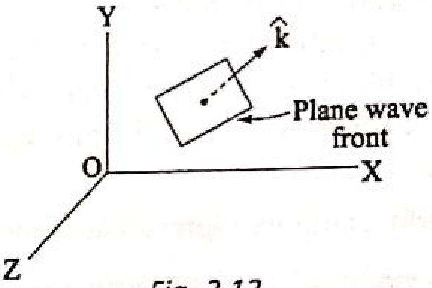

Fig. 2.12

2.7 Complex representation of a wave

Equation of a harmonic wave proceeding along positive X-axis can be written as

(i) $\psi(x,t) = a\cos(\omega t - kx)$

or (ii) $\psi(x,t) = a\sin(\omega t - kx)$

We know Euler formula : $e^{i\theta} = \cos\theta + i\sin\theta$. $\cos\theta$ is the real part and $\sin\theta$ is the imaginary part of $e^{i\theta}$.

Hence the above two wave equations can be represented as a *complex* quantity :

$$\psi(x,t) = ae^{i(\omega t - kx)} \quad \text{......................2.9}$$

The wave equation (i) is the *real part* of the complex quantity $\psi(x,t)$ and equation (ii) is the *imaginary part* of the complex quantity. Such representation is possible because in all algebraic manipulations real and imaginary parts are not mixed up; real parts and imaginary parts are added separately.

The *advantage* of writing the wave equation in this complex exponential form is that it easier to differentiate, integrate sum as series than sine and cosine functions. In particular we can *separate* the spatial and temporal part of complex representation as

$$\psi(x,t) = ae^{-ikx}e^{i\omega t}$$

This is not possible for sine and cosine functions. ae^{-ikx} is called complex amplitude (A) of the wave and $e^{i\omega t}$ is called the harmonic time factor. The intensity (I) of the wave can be found from the complex amplitude. It is the product of the amplitude and its complex conjugate :

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$I = AA^* = ae^{-ikx} \times ae^{+ikx} = a^2.$

When large number of waves superpose we require integration over the spatial part. Then the calculation of the resultant disturbance becomes very simple with the exponential functions. We shall see this when we shall study diffractions of light waves.

It should be clearly understood, however, that all physical quantities must be real. Once we get the final result we have to find either the real part or imaginary part of it to get the physical quantity of interest.

Similarly complex representation of the *plane* wave proceeding along the propagation vector $\vec{k}$ is

$$\psi(\vec{r},t) = ae^{i(\omega t - \vec{k}.\vec{r})} \quad ....2.10$$

2.8 Energy density of a wave

When wave passes through a medium energy is transmitted from one region to another. Hence there is energy distributed over the space occupied by the wave motion. We shall now calculate this energy contained per unit volume of the medium, which is called *energy density*.

For this purpose we consider a longitudinal wave passing through an elastic solid as discussed above. We consider a thin layer of small volume δV of the medium at distance x from the origin. The displacement of the layer at time t is given by the wave equation.

$$\therefore \text{ Displacement, } y = a\cos(\omega t - kx)$$

$$\text{Velocity, } v = \frac{\partial y}{\partial t} = -a\omega\sin(\omega t - kx)$$

If ρ is density of the medium, mass of the portion is $\rho\delta V$.

$\therefore$ Kinetic energy of this layer,

$$\delta E_k = \frac{1}{2}\rho\cdot\delta V\cdot v^2 = \frac{1}{2}\rho\cdot\delta V\cdot a^2\omega^2\sin^2(\omega t - kx)$$

$\therefore$ Kinetic energy per unit volume

$$u_k = \frac{1}{2}\rho a^2\omega^2\sin^2(\omega t - kx)$$

$$= \frac{1}{2}\rho a^2(2\pi n)^2\sin^2(\omega t - kx)$$

$$= 2\pi^2\rho n^2a^2\sin^2(\omega t - kx) \quad ....2.11$$

Potential energy in the layer is the *strain energy* produced in it as result of compression or rarefaction. Strain energy *per unit volume* is

$$u_{strain} = \frac{1}{2} \times \text{stress} \times \text{strain}.$$

From eqn.2.5, we can say that strain produced in the medium is $\frac{\partial y}{\partial x}$. From Hooke's law we can write, stress = $Y \times$ strain. Using these two results we get the potential energy per unit volume.

Potential energy per unit volume,

$$u_p = \frac{1}{2}Y\left(\frac{\partial y}{\partial x}\right)\times\left(\frac{\partial y}{\partial x}\right) = \frac{1}{2}Y\left(\frac{\partial y}{\partial x}\right)^2.$$

We know velocity of the wave, $c = \sqrt{\frac{Y}{\rho}}$ and

$$\frac{\partial y}{\partial x} = ak\sin(\omega t - kx).$$

$$\therefore u_p = \frac{1}{2}\rho c^2a^2k^2\sin^2(\omega t - kx).$$

$$= \frac{1}{2}\rho c^2a^2\left(\frac{2\pi}{\lambda}\right)^2\sin^2(\omega t - kx)$$

$$= 2\pi^2\rho n^2a^2\sin^2(\omega t - kx) \quad ....2.12$$

Therefore total energy density,

$$u = u_k + u_p = 4\pi^2\rho n^2a^2\sin^2(\omega t - kx) \quad ....2.13$$

From eqns. 2.11, 2.12 and 2.13 we can make the following observations:

(i) Kinetic and potential energy densities are *equal* in magnitude *at all instants* of time and are *in phase* with other. That means both reach the maximum value and zero value *at the same time*. Thus energy of a wave is *equally divided* between kinetic and potential forms.

The situation is different from simple harmonic vibration, where as potential energy decreases kinetic energy increases and vice versa. And total energy is constant.

(ii) Total energy density has *different values at different points at a particular instant*. The distribution energy at different points at a particular instant is shown in Fig.2.13.

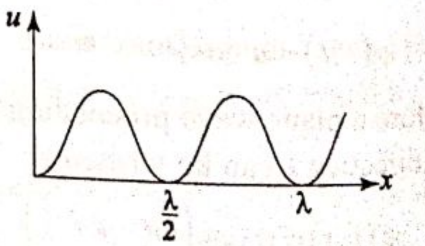

Fig. 2.13

Similarly, total energy density at a particular point also varies with time as the same square of a cosine function.

We know the average value of a square of a sine or cosine function over a time or space period is one-half.

∴ Average energy density is

$$\bar{u} = 2\pi^2 \rho n^2 a^2 \quad ....2.14$$

We see that *average energy density* is proportional to density of the medium ρ, the square of the frequency n, and square of the amplitude a.

(iii) We can write

$$u = 2\pi^2 \rho n^2 a^2 \,[1 - \cos 2(\omega t - kx)]$$

From this equation we can say that energy density *propagates like a wave.* The angular frequency is of this wave is 2ω, wavelength is λ/2, but its velocity is equal to that of displacement wave.

(iv) We see that energy density is proportional to density of the medium and to *squares* of frequency and amplitude.

All the above results also hold for transverse wave.

2.9 Intensity of a harmonic wave

When wave passes through a medium, energy transferred from one region to another region in the form of wave. Now we like to find the rate at which energy is transported by a wave. It is measured by intensity.

Intensity of a wave (I): Intensity of a wave at a point is the amount of average energy passing normally through unit area about that point.

We imagine a unit area A_1 about the point P oriented at right angle to an incoming plane wave,

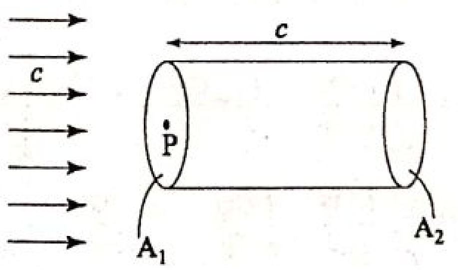

Fig. 2.14

Fig.2.14. If velocity of the wave is c, then the wavefront that strikes the area A_1 at the first instant of a second will arrive the surface A_2 at the last instant of the second. Hence all the energy that has crossed the surface A_1 in one second is contained in the cylinder of unit base and length c. Its volume is c.

∴ I = average energy density × c

$$= 2\pi^2 \rho n^2 a^2 \times c = 2\pi^2 \rho n^2 c a^2 \quad ....2.15$$

We see the different factors intensity of plane wave depends upon. In most situations, ρ, n and c remain unchanged and so we can say that intensity is proportional to the square of amplitude.

$\therefore I \propto a^2.$

In the case of sound waves in air, we can express intensity in terms of pressure amplitude (p_m) : We have $p_m = 2\pi c n \rho a$. Substituting value of amplitude (a), we get

Intensity, $I = 2\pi^2 \rho n^2 c a^2$

$$= 2\pi^2 \rho n^2 c \times \frac{P_m^2}{4\pi^2 c^2 n^2 \rho^2} = \frac{P_m^2}{2\rho c} \quad ....2.15a$$

2.10 Three dimensional wave

So far we have considered a plane wave which proceeds in a particular direction without spreading laterally. For a wave preceding along X-axis the corresponding wave function $\psi(x,t)$ satisfies the wave equation :

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}$$

The solution is of the form :

$$\psi(x,t) = a \cos(\omega t - kx).$$

Now we consider waves which spread in *all directions* in the three dimensional space. For example, sound waves, light waves, etc.

We suppose that the medium is *homogeneous* and *isotropic*. Therefore there is nothing to distinguish one Cartesian coordinate from the other two. Hence all the coordinates (x, y, z) should occur *symmetrically* in wave equation. We can, therefore, easily generalise the above equation and get the differential equation for a wave proceeding in all directions as given by

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \quad ....2.16$$

In a compact notation $\nabla^2 \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$ 2.16a

Its solution is given by $\psi(\vec{r},t) = a\cos(\omega t - \vec{k}\cdot\vec{r})$

We notice the same eqn. 2.8 represents a plane wave. The difference is that for the plane wave

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the propagation vector $\vec{k}$ has the same direction at all points (x, y, z), but for a wave proceeding in all directions $\vec{k}$ has different directions at different points.

2.10.1 *Spherical wave*

Now we shall study a very important kind of wave in the three dimensional space.

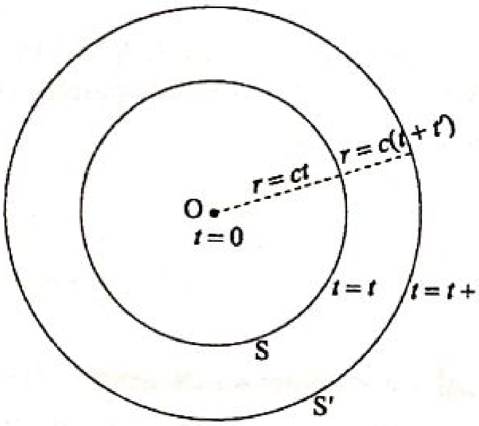

Fig. 2.15

We imagine a point source O producing a wave which spreads in all directions, Fig.2.15. If the medium is homogeneous and isotropic, the velocity of the wave has the same value everywhere and in every direction. Hence the disturbance produced at O at time $t = 0$ will arrive at each point of the surface of the sphere S of radius $r = ct$ at time $t = t$. Since the waves have traversed equal distance and taken equal time, the waves arrive with the same phase. Therefore the surface of the sphere is locus of points having the same phase of vibration. By definition the surface of the sphere is the *spherical wavefront* of the wave. At a later time $t + t'$, the sphere S' of radius $r = c(t + t')$ is the wavefront.

Let us first see how intensity of the wave changes with distance r from the source O. Suppose that the source emits energy E per second. Then E amount of energy will pass through the surface of the sphere per second. This energy is passing normally (radially) through the surface of total area $A = 4\pi r^2$. Therefore amount of energy passing normally through unit area per unit time, i.e., intensity is given by

$$I = \frac{E}{A} = \frac{E}{4\pi r^2} \;\therefore I \propto \frac{1}{r^2}.$$

Intensity at a point varies inversely as the square of the distance from the source. This result is quite expected, because the energy is spreading over bigger and bigger areas. In contrast intensity of a plane wave does not change with distance as it does not spread laterally.

Now we shall find the differential equation of a spherical wave and its solution.

For spherical wave more convenient coordinates is spherical polar coordinates (r, θ, φ) than the rectangular coordinates (x, y, z), Fig.2.16,

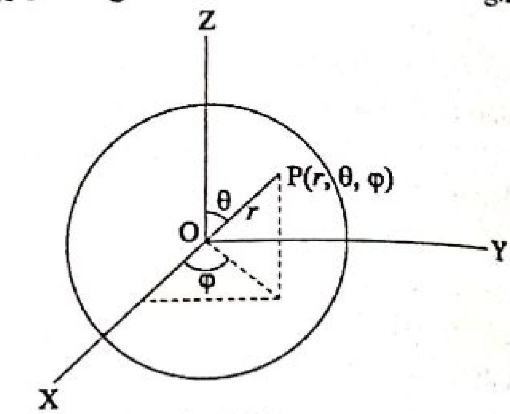

Fig. 2.16

as there is spherical symmetry. As a result the wave function ψ should not depend on the angles θ and φ. We have

$$\psi(r,t) = \psi(r,\theta,\varphi,t) = \psi(r,t)$$

Let us see how wave equation becomes simpler with the above condition.

$$r^2 = x^2 + y^2 + z^2. \;\therefore\; 2r\frac{\partial r}{\partial x} = 2x \;.\therefore\; \frac{\partial r}{\partial x} = \frac{x}{r}.$$

$$\therefore \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial r}\frac{\partial r}{\partial x} = \frac{x}{r}\frac{\partial \psi}{\partial r}.$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r}\frac{\partial \psi}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{x}{r}\frac{\partial \psi}{\partial r}\right)\frac{\partial r}{\partial x}$$

$$= \left[\frac{1}{r}\frac{r}{x}\frac{\partial \psi}{\partial r} - \frac{1}{r^2}x\frac{\partial \psi}{\partial r} + \frac{x}{r}\frac{\partial^2 \psi}{\partial r^2}\right]\frac{x}{r}$$

$$= \frac{x^2}{r^2}\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{x^2}{r^3}\frac{\partial \psi}{\partial r}.$$

The similar relations hold for the y and z coordinates.

Substituting all these in eqn.2.16, we get

$$\frac{x^2 + y^2 + z^2}{r^2}\frac{\partial^2 \psi}{\partial r^2} + \frac{3}{r}\frac{\partial \psi}{\partial r} - \frac{x^2 + y^2 + z^2}{r^3}\frac{\partial \psi}{\partial r} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

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$$\text{or, } \frac{\partial^2 \psi}{\partial r^2}+\frac{2}{r}\frac{\partial \psi}{\partial r}=\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

Multiplying both sides by r we get the equation of a spherical wave as

$$r\frac{\partial^2 \psi}{\partial r^2}+2\frac{\partial \psi}{\partial r}=r\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \quad \text{.................(i)}$$

Now $\frac{\partial^2}{\partial r^2}(r\psi)=\frac{\partial}{\partial r}\left(\psi+r\frac{\partial \psi}{\partial r}\right)= r\frac{\partial^2 \psi}{\partial r^2}+2\frac{\partial \psi}{\partial r}$

and r does not depend on time.

Hence the above eqn.(i) can be written as

$$\frac{\partial^2}{\partial r^2}(r\psi)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(r\psi)$$

It is equation of one-dimensional wave equation and its general solution is

$$r\psi(r,t)=f(ct-r)+g(ct+r)$$

∴ Wave function is

$$\psi(r,t)=\frac{1}{r}f(ct-r)+\frac{1}{r}g(ct+r)$$

The first term in right hand side of the above equation represents the spherical wave spreading radially outward from the source at the centre. The second term represents the spherical wave converging toward the centre.

For a harmonic spherical wave corresponding to the first term is given by

$$\psi(r,t)=\frac{a}{r}\cos(\omega t-kr)$$

We see that the amplitude of the spherical wave decreases inversely with r, which is the distance from the source. As intensity is proportional to square of the amplitude, intensity falls off as $1/r^2$, the same result we get earlier on physical considerations.

2.10.2 *Cylindrical wave*

For a point source or very small source of light we get spherical wave. If the point source is at a very large distance or at the focus of a convex lens we get a plane wavefront. But if there is line source placed behind a thin slit we shall get a cylindrical wavefront with its axis coinciding with the line source. If a plane wavefront strikes a slit from behind the wavefront emerging on the other side is cylindrical. Though in many experiments we find such cylindrical wavefronts, we shall not discuss about such waves, we treat these as plane waves.

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