PHY-H-CC-T-03: ELECTRICY AND MAGNETISM

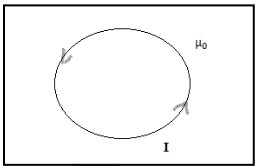
LECTURE-3 (Pabitra Halder (PH), Department of Physics, Berhampore Girls' College)

Ampere's circuital theorem:

Statement: The amount of work done in carrying a unit positive magnetic pole once around an electrical circuit (where a current I flows) is equal to the product of current (I) and the permeability of free space(μ_0).

It may be again stated as, the line integral of magnetic induction ($\vec{B}$) over any closed path round a closed electric circuit carrying a current of I is equal to the product of μ_0 and I.

If $\vec{B}$ be the magnetic induction at any elementary line element $\overrightarrow{dl}$ of the closed loop L.



$\oint \vec{B} \cdot \overrightarrow{dl} = \iint (\vec{\nabla} \times \vec{B}) . \overrightarrow{ds}$ →From Stoke's theorem.

Where S is the area enclosed by L.

$\oint \vec{B} \cdot \overrightarrow{dl} = \iint \mu_0 \vec{J} \cdot \overrightarrow{ds} = \mu_0 \iint \vec{J} \cdot \overrightarrow{ds}$

$\oint \vec{B} \cdot \overrightarrow{dl} = \mu_0$ I

Hence, Ampere's circuital theorem is established.

Application:

1. Magnetic field due to a long Solenoid of length L and number of turns N.

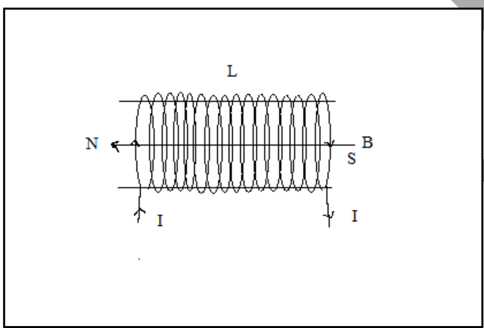


Fig.1 Long Solenoid

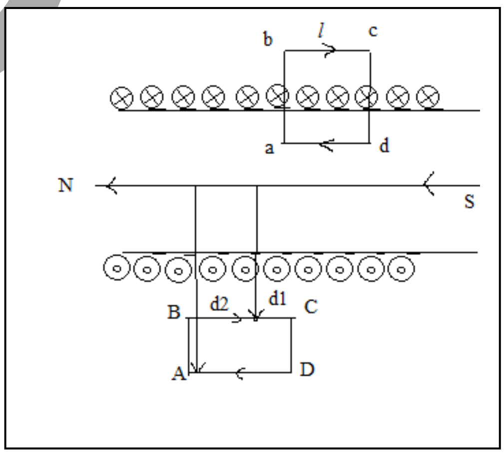


Fig. 2 Cross-section of Solenoid

We take the solenoid to be closely wound so that each turn can be considered to be circular. We can prove that the field due to such a solenoid is entirely confined to its interior, i.e. the field outside is zero.

Outside the Solenoid:

To see this consider a rectangular amperian loop ABCD parallel to the axis of the solenoid.

Field everywhere on BC is constant and is B(d1) . Likewise the field everywhere on DA is (d2) . By Righthand rule, the field on DA is directed along the loop while that on BC is oppositely directed. On the sides AB and CD, the magnetic field direction is perpendicular to the length element and hence is zero everywhere on these two sides. Thus

$\oint \vec{B} \cdot \overrightarrow{dl} = l[B(d2) - B(d1)]$

By Ampere's law, the value of the integral is zero as no current is enclosed by the loop.

Thus, $B(d2) = B(d1)$. The field outside the solenoid is, therefore, independent of the distance from the axis of the solenoid. However, from physical point of view, we expect the field to vanish at large distances. Thus, $B(d2) = B(d1) = 0$.

Inside the Solenoid:

To find the field inside, take an amperian loop abcd with its length parallel to the axis as before, but with one of the sides inside the solenoid while the other is outside. The only contribution to $\oint \vec{B} \cdot \overrightarrow{dl}$ come from the side (da) is

$\oint \vec{B} \cdot \overrightarrow{dl} = Bl = \mu_0 I_{enclosed} = \mu_0 n I l$

Where, I is the current through each turn and 'n' is number of turns per unit length. $I_{enclosed} = nIl$

because the number of turns threading the loop is nl.

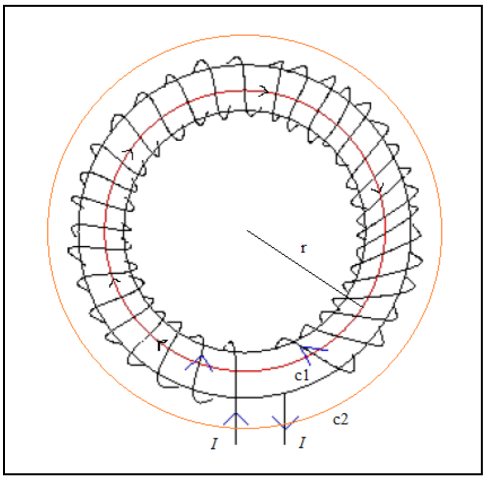
Hence, $B = \mu_0 nI$ → independent of the distance from the axis.

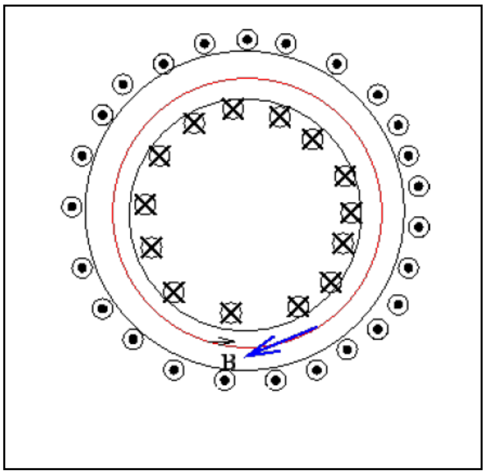
Special case:

It can be shown that the magnetic field at the end of a long solenoid will be

$$B = \frac{1}{2}\mu_0 nI.$$

2. Magnetic field due to a toroid:





The configuration in which we have closely spaced windings of a wire wound around a ring is called a toroid. Let a current I flow through the windings.

Inside the toroid:

Consider a circular path c1 inside the ring and concentric with it. The symmetry shows that the magnetic field is constant everywhere along the path c1 and tangential to it. If the current direction is as shown in figure, the direction of the line of constant B is clock-wise. The total current enclosed by the path c1 is NI, where N is the total number of turns on the toroid. Ampere's circuital law for the path c1 gives

$\oint \vec{B} \cdot \vec{dl} = \mu_0 NI \rightarrow B\,(2\pi r) = \mu_0 NI$, where r is the radius of the circular path c1. Thus

$$B = \frac{\mu_0 NI}{2\pi r} \quad \cdots\cdots\cdots (1)$$

If the thickness of the toroid is much less than its mean radius R, then $r \approx R$ and equation (1) gives the field everywhere inside the toroid. As $N/2\pi R = n$ gives the number of turns per unit length of the toroid, we can write,

$$B = \mu_0 nI$$

Outside the toroid:

Considering the circular path c2 we get from Ampere's circuital law

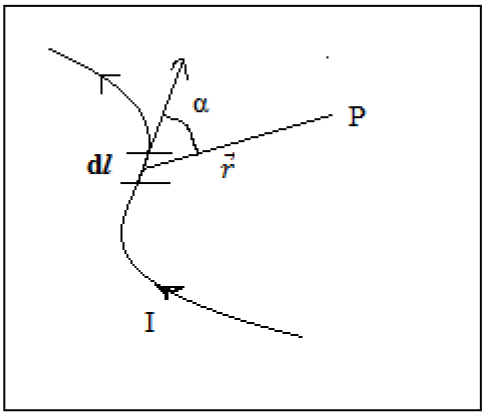
$\oint \vec{B} \cdot \vec{dl} = \mu_0 N \times 0 \rightarrow B = 0$, as the net current enclosed by the path c2 is zero.

Limitation:

Ampere's law is always true, but is only a useful tool to evaluate the magnetic field if the symmetry of the system enables you to pull outside the line integral. The configurations that can be handled by Ampere's law are: 1. Infinite straight lines, 2. Infinite planes, 3. Infinite solenoids, 4. Toroids.

Biot-Savart Law:

If the magnetic induction at a point due to a line element $\vec{dl}$ of a conductor carrying a current I ampere be given by $d\vec{B}$, the law states that



i) $|d\vec{B}| \propto I$, ii) $|d\vec{B}| \propto dl$, iii) $|d\vec{B}| \propto \sin\alpha$ and iv) $|d\vec{B}| \propto \frac{1}{r^2}$

Vectorically, $d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,\vec{dl} \times \vec{r}}{r^3}$, where μ_0 is the permeability of free space.

The magnetic induction $d\vec{B}$ is perpendicular to the plane containing

the line element $\vec{dl}$ and the position vector $\vec{r}$.

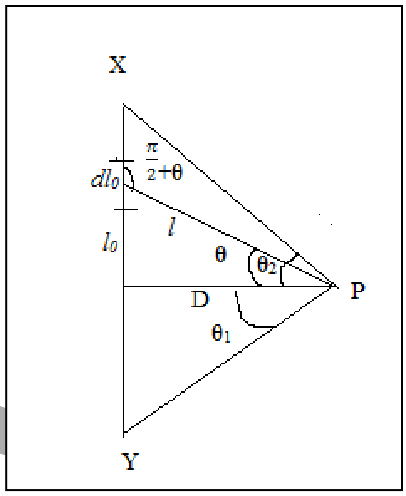
In absence or presence of a magnetic material the magnetic induction at a point $\vec{r}$ due to a closed conductor is given by

$$\vec{B} = \frac{\mu_0}{4\pi} \oint \frac{I\,\vec{dl} \times \vec{r}}{r^3}$$

<u>Applications of Biot-savart Law:</u>

1. Magnetic field due to the current flowing through a straight wire of finite length:

Let XY be a straight-line conductor carrying a current of *I* amp and P be a point at a distance D from XY. The magnetic induction $\vec{B}$ at P is required. From Biot-Savart law, the magnetic induction at P due to element dl_0 of the wire is



$$dB = \frac{\mu_0}{4\pi} \frac{I\,dl\,\sin(\frac{\pi}{2}+\theta)}{l^2} = \frac{\mu_0}{4\pi} \frac{I\,dl\,\cos\theta}{l^2}$$

Where *l* is the length of the radius vector of the point P from *dl*,

Now dl= D $sec^2\theta$ dθ, also l= D sec θ

$$dB = \frac{\mu_0 I}{4\pi D} \cos\theta \, d\theta$$

Therefore, the total magnetic induction due to the entire wire is

$$B = \frac{\mu_0 I}{4\pi D} \int_{-\theta 1}^{\theta 2} \cos\theta \, d\theta = \frac{\mu_0 I}{4\pi D} (\sin\theta 1 + \sin\theta 2) \text{ wb/m}^2.$$

The direction of $\vec{B}$ is perpendicular to the plane of the paper and into it.

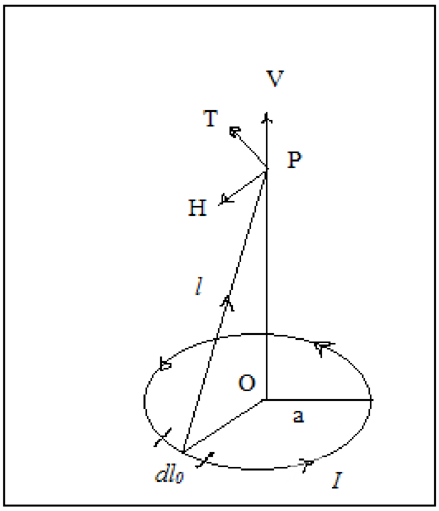
Special case:

For an infinite wire $\theta_1 = \theta_2 = \frac{\pi}{2}$

$$|\vec{B}| = \frac{\mu_0 I}{2\pi D}.$$

2. Magnetic field at a point on the axis of a circular conductor carrying a current:

Let us consider a circular conductor carrying a current I. Now the magnetic Field at a point P along the axis of the loop at a distance z is to be determined.



Consider an element dl_0 on the loop. From Biot-Savart law the magnetic field at a point P due to the element $\overrightarrow{dl_0}$ is

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\,\overrightarrow{dl_0} \times \vec{l}}{l^3}$$, where μ_0 is the permeability of free space.

Where l is the distance of the point P from element dl_0 on the loop. Since, the angle

Between $\overrightarrow{dl_0}$ and $\vec{l}$ is $\frac{\pi}{2}$, we have

$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{dl0}{l^2}$ along PT

Because of symmetry, when entire loop is considered, the component perpendicular to the axis cancel out and only components along the axis contribute.

Now, PV= PT sin$< HPT$ = PT (a/l), a is the radius of the loop. Hence the component of $d\vec{B}$ along the loop axis is

$(d\vec{B})_{axis} = \frac{\mu_0 I}{4\pi} \frac{dl0}{l^2}(a/l) = \frac{\mu_0 I a}{4\pi l^3}\, dl_0$

The magnetic field due to the entire loop is

$(\vec{B})_{axis} = \frac{\mu_0 I a}{4\pi l^3} \int dl_0 = \frac{\mu_0 I a}{4\pi l^3}(2\pi a) = \frac{\mu_0 I a^2}{2l^3} = \frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}}$

If coil has N number of closely wound turns, then the magnetic field

$(\vec{B})_{axis} = \frac{\mu_0 N I a^2}{2(a^2+z^2)^{3/2}}$ ....(1)

Special case:

The field at the centre of the loop is obtain by putting z=0 in equation (1). The field is

$(\vec{B})_{centre} = \frac{\mu_0 I}{2a}$.

Problems:

1. Find the magnetic field at a point P for each of the steady current configurations shown in figure below.

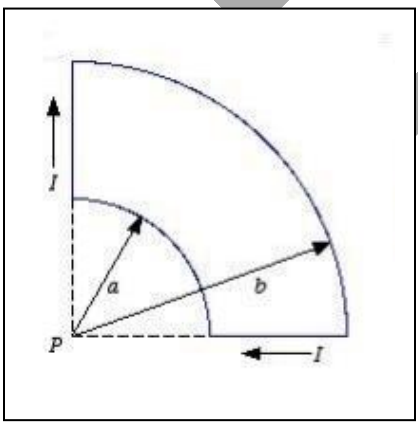


Fig.1

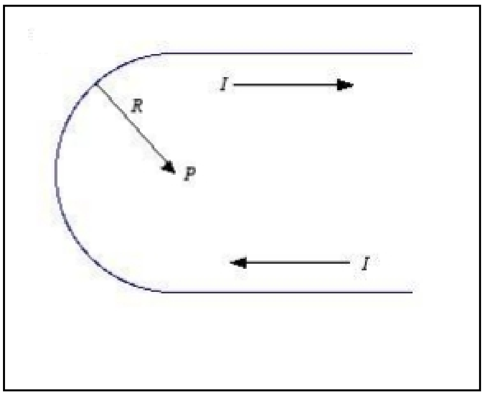


Fig.2

First Case:

The straight segments produced no field at P. We know that the magnetic field at the centre of circular loop of radius a carrying a current I is $(\vec{B})_{\text{centre}} = \frac{\mu_0 I}{2a}$.

Here, P is the centre of two concentric quarter-circles of radius ‘a’ and ‘b’ respectively. Then the magnetic field at P due to the quarter-circle of radius ‘a’ is

$\overrightarrow{B_1} = \frac{1}{4} \times \frac{\mu_0 I}{2a} \longrightarrow$ outward normal to the page.

Similarly, the magnetic field at P due to the quarter-circle of radius ‘b’ is

$\overrightarrow{B_2} = \frac{1}{4} \times \frac{\mu_0 I}{2b} \longrightarrow$ Inward normal to the page.

Since, b>a then the magnetic field at P due to the entire current configuration is

$(\vec{B})_P = \overrightarrow{B_1} - \overrightarrow{B_2} = \frac{\mu_0 I}{8}\left(\frac{1}{a} - \frac{1}{b}\right) \longrightarrow$ outward normal to the page.

Second Case:

The magnetic field at P due to the two half-lines are the same as one infinite line

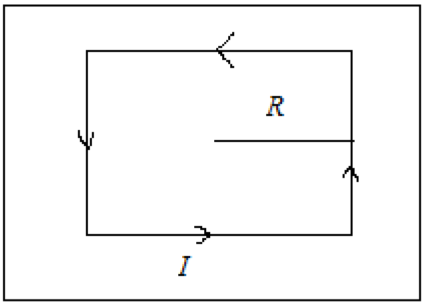
$\overrightarrow{B_1} = \frac{\mu_0 I}{2\pi R} \longrightarrow$ Inward normal to the page.

The magnetic field at P due to the half-circle of radius R is

$\overrightarrow{B_2} = \frac{1}{2} \times \frac{\mu_0 I}{2R} \longrightarrow$ Inward normal to the page.

The total magnetic field at P due to entire current configuration is

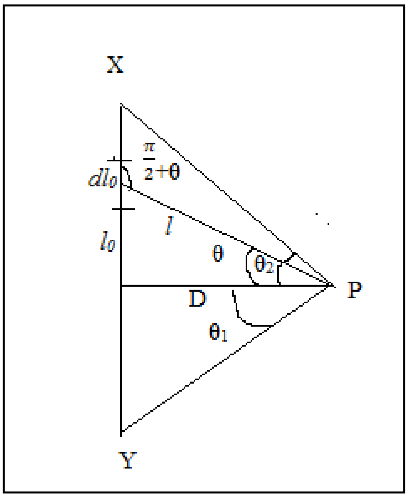
$(\vec{B})_P = \overrightarrow{B_1} + \overrightarrow{B_2} = \frac{\mu_0 I}{4R}\left(1 + \frac{2}{\pi}\right) \longrightarrow$ Inward normal to the page.



2. (a) Find the magnetic field at the centre of a square loop, which carries a steady current *I*. Let R be the distance from centre to side.

(b) Find the field at the centre of a regular n-sided polygon, carrying a steady current *I*. Again, let R be the distance from centre to any side.

(c) Check that your formula reduced to the field at the centre of a circular loop, in the limit n→∞.



Ans. (a) The magnetic field at P due to the finite straight I amount current carrying wire be $\vec{B} = \frac{\mu_0 I}{4\pi D}(\sin\theta 1 + \sin\theta 2) \longrightarrow$ Inward normal to the page.

Here, D=R and θ2= - θ1 = 45^0

The magnetic field at the centre of square loop due to the one arm is

$\overrightarrow{B_1} = \frac{\mu_0 I}{4\pi R}$ (sin 45 +sin 45) = $\frac{\mu_0 I}{4\pi R}\sqrt{2} \rightarrow$ outward normal to the page.

The magnetic field at the centre of square loop is

$\vec{B} = 4\times \overrightarrow{B_1} = \frac{\mu_0 I}{\pi R}\sqrt{2} \rightarrow$ outward normal to the page.

(b) Here, D=R and θ2= - θ1 = $\frac{\pi}{n}$ for n-sided polygon

The total magnetic field at the centre of a n-sided polygon is

$\vec{B} = n\times \frac{\mu_0 I}{4\pi R}(\sin\frac{\pi}{n} + \sin\frac{\pi}{n}) = \frac{\mu_0 n I}{2\pi R}\sin\frac{\pi}{n} \rightarrow$ outward normal to the page.

(c) As n→∞, θ will be very small. So, sinθ ≈ θ.

Then the magnetic field at the centre of a n-sided polygon as n→∞ is

$\vec{B} = \frac{\mu_0 n I}{2\pi R}\times\frac{\pi}{n} = \frac{\mu_0 I}{2R} \rightarrow$ magnetic field at the centre of a circular loop of radius R carrying a current *I*.

3. (a) Find the force on a square loop placed as shown in fig.1, near an infinite straight wire. Both the loop and wire carry a steady current *I*.

(b) Find the force on a triangular loop placed as shown in fig.2, near an infinite straight wire. Both the loop and wire carry a steady current *I*.

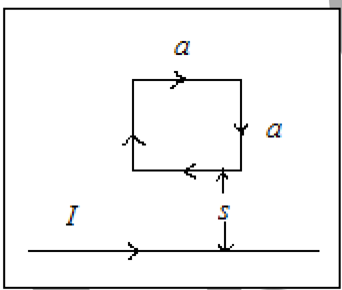


Fig.1

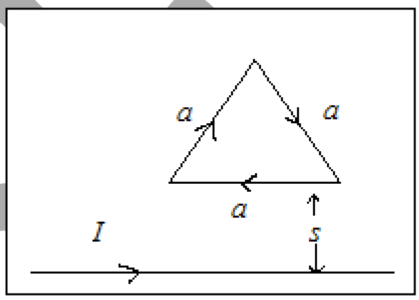


Fig.2

Ans.(a) The forces on two sides of square loop, normal to the infinite current carrying wire cancel.

The force on the bottom side of the square loop is

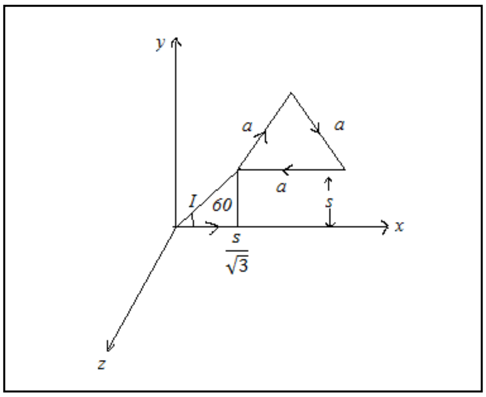
$\overrightarrow{F_1} = I(\vec{l}\times\vec{B}) = \left(\frac{\mu_0 I}{2\pi s}\right)\times I a = \frac{\mu_0 I^2 a}{2\pi s} \rightarrow$ upward (magnetic field due to the infinite straight wire $\vec{B} = \frac{\mu_0 I}{2\pi s}$).

Similarly, the force on the top side of the square loop is

$\overrightarrow{F_2} = \frac{\mu_0 I^2 a}{2\pi(s+a)} \rightarrow$ downward (magnetic field due to the infinite straight wire $\vec{B} = \frac{\mu_0 I}{2\pi s}$).

Therefore, the net force on the square loop due to the infinite straight wire carrying a current *I* is

$\vec{F} = \overrightarrow{F_1} - \overrightarrow{F_2} = \frac{\mu_0 I^2 a}{2\pi s} - \frac{\mu_0 I^2 a}{2\pi(s+a)} = \frac{\mu_0 I^2 a^2}{2\pi s(s+a)} \rightarrow$ upward.



(b) The force on the bottom side of the triangular loop is

$\overrightarrow{F_1} = \frac{\mu_0 I^2 a}{2\pi s} \rightarrow$ upward (magnetic field due to the infinite straight wire $\vec{B} = \frac{\mu_0 I}{2\pi s}$).

On the left side, the magnetic field $\vec{B} = \frac{\mu_0 I}{2\pi y}\,\hat{z}$;

The elementary magnetic force on the left side of the triangular loop is

$d\vec{F} = I\,(\overrightarrow{dl} \times \vec{B}) = I\,(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}) \times \frac{\mu_0 I}{2\pi y}\,\hat{z} = \frac{\mu_0 I^2}{2\pi y}\,(-\,dx\,\hat{y} + dy\,\hat{x})$

But the x-component cancels the corresponding term from the right side and $F_y = -\frac{\mu_0 I^2}{2\pi}\int_{\frac{s}{\sqrt{3}}}^{\boxed{\frac{s}{\sqrt{3}}+\frac{a}{2}}} \frac{1}{y}\,dy$

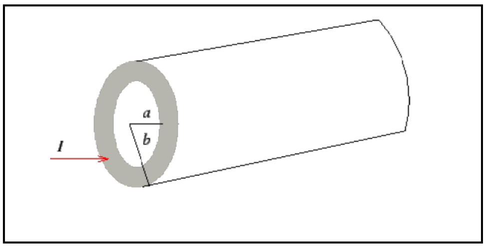
Here $y = \sqrt{3}\,x$, so $F_y = -\frac{\mu_0 I^2}{2\sqrt{3}\pi}\ln\left(\frac{\frac{s}{\sqrt{3}}+a/2}{s/\sqrt{3}}\right) = -\frac{\mu_0 I^2}{2\sqrt{3}\pi}\ln\left(1+\frac{\sqrt{3}a}{2s}\right)$

Similarly, the force on the right side of the triangle is the same as left side, so the net force on the triangle is

$\vec{F} = \frac{\mu_0 I^2}{2\pi}\left[\frac{a}{s} - \frac{2}{\sqrt{3}}\ln\left(1+\frac{\sqrt{3}a}{2s}\right)\right] \rightarrow$ upward.

Exercise:

1. A hollow cylindrical conductor of infinite length carries uniformly distributed current I from $a < r < b$. Determine the magnetic field for all r.



2. A long wire of cross-sectional radius R carries a current I. The current density varies as the square of the distance from the axis of the wire. Find the magnetic field for $r<R$ and for $r>R$.