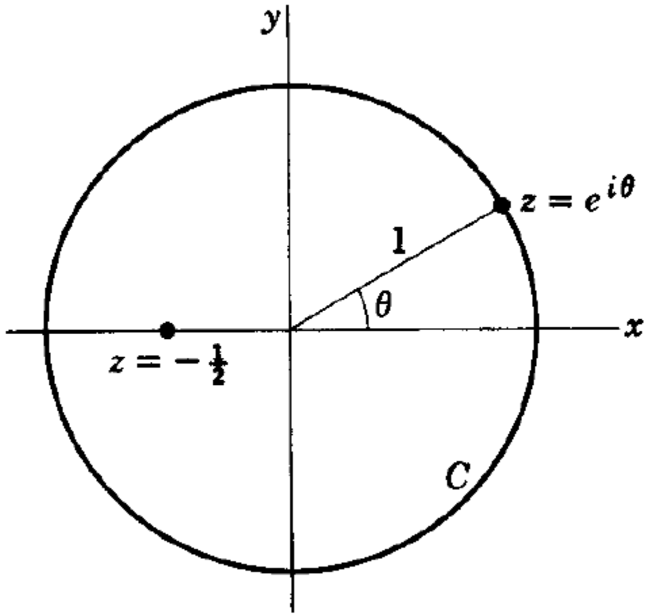
Definite Integrals using Residue Theorem

Acknowledgement

- Mathematical Methods in the Physical Sciences – Mary L. Boas

$$\text{Find } I = \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$$



$$z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\, d\theta = iz\, d\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}$$

$$I = \int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$$

$$d\theta = \frac{1}{iz}\, dz$$

$$\frac{z + \frac{1}{z}}{2}$$

$$I = \oint_C \frac{\frac{1}{iz}\, dz}{5 + 2(z + 1/z)}$$

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$$= \frac{1}{i} \oint_C \frac{dz}{5z + 2z^2 + 2}$$

$$= \frac{1}{i} \oint_C \frac{dz}{(2z + 1)(z + 2)}$$

C is the unit circle

The integrand has poles at $z = -\frac{1}{2}$ and $z = -2$

$z = -\frac{1}{2}$ is inside the contour C.

The residue of $1/[(2z+1)(z+2)]$ at $z = -\frac{1}{2}$ is

$$R(-\tfrac{1}{2}) = \lim_{z\to -1/2} (z+\tfrac{1}{2}) \cdot \frac{1}{(2z+1)(z+2)} = \frac{1}{2(z+2)}\bigg|_{z=-1/2}$$
$$= \tfrac{1}{3}$$

Then by the residue theorem

$$I = \frac{1}{i}\, 2\pi i R(-\tfrac{1}{2}) = 2\pi \cdot \tfrac{1}{3} = \frac{2\pi}{3}$$

This method can be used to evaluate the integral of any rational function of $\sin\theta$ and $\cos\theta$ between 0 and 2π, provided the denominator is never zero for any value of θ. You can also find an integral from 0 to π if the integrand is even, since the integral from 0 to 2π of an even periodic function is twice the integral from 0 to π of the same function.

Evaluate the integrals by using the residue theorem

1. $$\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta}$$

2. $$\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$$

3. $$\int_0^{2\pi} \frac{d\theta}{5 - 4\sin\theta}$$

4. $$\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{5 + 3\cos\theta}$$

5. $$\int_0^{\pi} \frac{d\theta}{1 - 2r\cos\theta + r^2} \qquad (0 \leq r < 1)$$

6. $$\int_0^{\pi} \frac{d\theta}{(2 + \cos\theta)^2}$$

7. $$\int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{5 + 4\cos\theta}$$

8. $$\int_0^{\pi} \frac{\sin^2\theta \, d\theta}{13 - 12\cos\theta}$$