5.8 Velocity of Sound in Gases

When a sound wave propagates in a gas, the pressure changes, i.e., the compressions and rarefactions occur so rapidly that there is no exchange of heat between the layers and the surroundings. The process is thus *adiabatic* obeying the equation PV^γ = constant, where γ is the ratio of the specific heat of the gas at constant pressure to that at constant volume. The differentiation of the above equation gives

$$\gamma P V^{\gamma-1} dV + V^\gamma dP = 0$$

$$\text{or,} \quad \gamma P dV = -V dP$$

$$\text{or,} \quad \frac{dP}{dV} = -\frac{\gamma P}{V}. \tag{5.28}$$

The bulk modulus is

$$K = -\frac{p}{\frac{\delta V}{V_0}} = -V\frac{dP}{dV} \tag{5.29}$$

since $p = dP$, $\delta V = dV$ and $V_0 = V$. Substituting for $\frac{dP}{dV}$ from Eq. (5.28) into Eq. (5.29) gives $K = \gamma P$, so that the velocity of sound in the gas is

$$c = \sqrt{\frac{K}{\rho_0}} = \sqrt{\frac{\gamma P}{\rho}}. \tag{5.30}$$

For air at NTP, $\rho = 1.293$ kg/m^3, $P = 1.013 \times 10^5$ Pa and $\gamma = 1.41$. Substituting these values in Eq. (5.30), we have $c = 332.3$ m/s. in good

5.10 Velocity of Longitudinal Waves in a Solid Bar

We consider a solid bar of cross sectional area α, the length of the bar being comparable to the wavelength of the longitudinal wave propagating along the bar. The bar is assumed to be thin enough so that a longitudinal stress produces the same displacement of the molecules over a given cross section. Let the density and Young's modulus of the material of the bar be ρ and Y, respectively. We choose two planes A_1 and B_1 inside the bar perpendicular to its axis, the initial distances of A_1 and B_1 from an arbitrary origin being x and $x + \delta x$, respectively (Fig. 5.4). The passage of the longitudinal wave causes the molecules in the rod to be displaced, thus shifting the planes A_1 and B_1. Let at any instant of time, the plane A_1 be displaced by ξ to the position A_2. Correspondingly the plane B_1 is displaced by $\xi + \delta\xi$ to B_2. The distance A_2B_2 is $\delta x + \delta\xi = \delta x + \frac{\partial \xi}{\partial x}\delta x$. Thus the distance between the planes increases by $\delta\xi = \frac{\partial \xi}{\partial x}\delta x$. Therefore, the elongation of the bar per unit length at this point is $\frac{\partial \xi}{\partial x}$. Consequently, the compressive strain is $-\frac{\partial \xi}{\partial x}$.

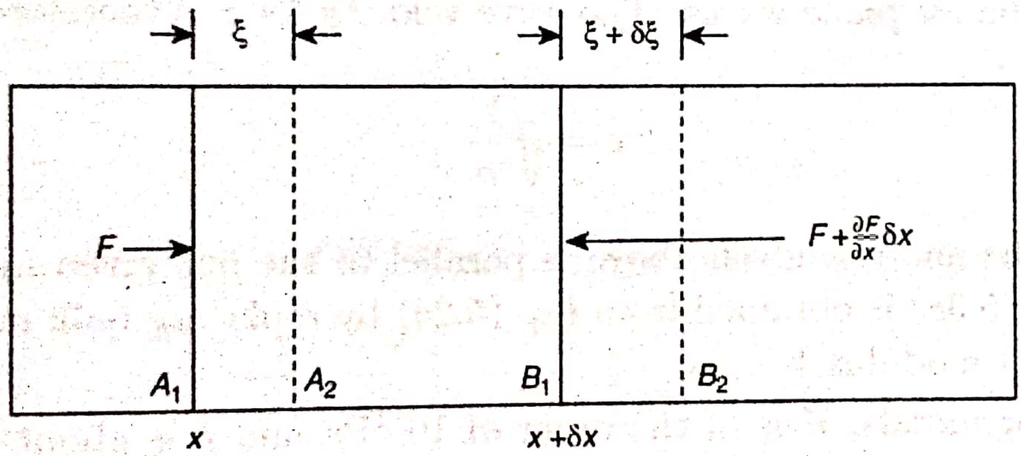

Fig. 5.4 Propagation of longitudinal waves in a thin solid rod

Since $\xi \ll \delta x$, the forces on A_1 and A_2 and those on B_1 and B_2 can be taken to be equal. If F is the force on A_1 (or A_2) acting in the positive x-direction, the force on B_1 (or B_2) acting in the opposite direction is $F + \frac{\partial F}{\partial x}\delta x$. These two forces produce two effects:

(i) A compressive stress F/α on the slab A_1B_1 develops the compressive strain $-\frac{\partial\xi}{\partial x}$. From Hooke's law, we have

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{F/\alpha}{\left(-\frac{\partial\xi}{\partial x}\right)}$$

$$\text{or,} \quad F = -Y\alpha\frac{\partial\xi}{\partial x}. \tag{5.34}$$

(ii) An unbalanced resultant force $\frac{\partial F}{\partial x}\delta x$ causes, by Newton's second law of motion, an acceleration $\frac{\partial^2\xi}{\partial t^2}$ of the slab of mass $\rho\alpha\delta x$. Since Force = mass × acceleration, we have

$$-\frac{\partial F}{\partial x}\delta x = \rho\alpha\delta x\frac{\partial^2\xi}{\partial t^2}, \tag{5.35}$$

the negative sign implying that the force is in the negative x-direction. Equation (5.35) simplifies to

$$-\frac{\partial F}{\partial x} = \rho\alpha\frac{\partial^2\xi}{\partial t^2}. \tag{5.36}$$

Differentiating Eq. (5.34) with respect to x and substituting in Eq. (5.36), we obtain

$$Y\frac{\partial^2\xi}{\partial x^2} = \rho\frac{\partial^2\xi}{\partial t^2}$$

$$\text{or,} \quad \frac{\partial^2\xi}{\partial t^2} = \frac{Y}{\rho}\frac{\partial^2\xi}{\partial x^2}. \tag{5.37}$$

Comparison of Eq. (5.37) with Eq. (5.8) shows that ξ satisfies the differential wave equation for plane waves. The wave velocity for ξ is therefore given by

$$c = \sqrt{\frac{Y}{\rho}}. \tag{5.38}$$

Note that the analysis given above is parallel to the one given in Sec. 5.7. In fact, Eq. (5.38) is obtained from Eq. (5.24) by replacing bulk modulus K with Young's modulus Y.

Since, for metals, Y is of the order of 10^{11}Pa and ρ is about 10^4kg/m^3, the velocity of sound in thin metal rods is a few times 10^3 m/s. This velocity is greater than that in liquids or gases.

Observations

If, instead of in a long thin bar, longitudinal waves propagate in an extended solid medium, there would be no lateral expansion or contraction associated with the longitudinal contraction or expansion during the passage of the wave, as we had for a long thin bar. Therefore, the axial modulus X should replace Young's modulus Y in Eqs. (5.37) and (5.38), so that in an extended solid the velocity of sound is given by

$$c = \sqrt{\frac{X}{\rho}}. \tag{5.39}$$

The axial modulus X can be written as

$$X = \frac{3K + 4n}{3} = \frac{Y(1-\sigma)}{(1+\sigma)(1-2\sigma)} \tag{5.40}$$

where K is the bulk modulus, n is the rigidity modulus, and σ is Poisson's ratio. So, Eq. (5.39) gives

$$c = \sqrt{\frac{3K + 4n}{3\rho}} = \sqrt{\frac{Y(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)}}. \tag{5.41}$$

Waves travelling through the earth's crust during an earthquake are approximate examples of waves travelling in an extended solid.