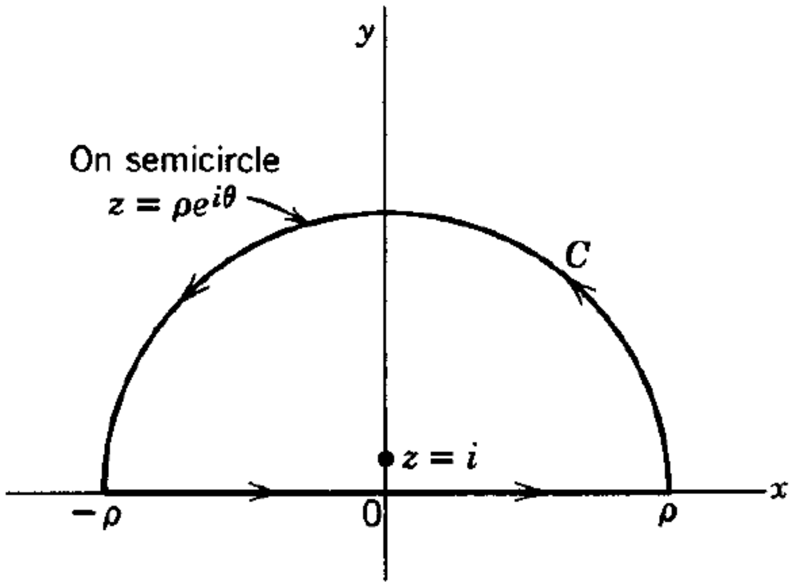
Definite Integrals using Residue Theorem-2

Acknowledgement

- Mathematical Methods in the Physical Sciences – Mary L. Boas

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

We consider $\oint_C \frac{dz}{1+z^2}$



where C is the closed boundary of the semicircle For any $\rho > 1$, the semicircle incloses the singular point $z = i$ and no others; the residue of the integrand at $z = i$ is

$$R(i) = \lim_{z \to i} (z - i) \frac{1}{(z - i)(z + i)} = \frac{1}{2i}$$

Then the value of the contour integral is $2\pi i(1/2i) = \pi$.

Let us write the integral in two parts:

(1) an integral along the x axis from $-\rho$ to ρ; for this part $z = x$

(2) an integral along the semicircle, where $z = \rho e^{i\theta}$

$$\int_C \frac{dz}{1+z^2} = \int_{-\rho}^{\rho} \frac{dx}{1+x^2} + \int_0^{\pi} \frac{\rho i e^{i\theta}\, d\theta}{1+\rho^2 e^{2i\theta}}$$

(1) (2)

We know that the value of the contour integral is π no matter how large ρ becomes since there are no other singular points besides $z = i$ in the upper half-plane. Let $\rho \to \infty$; then the second integral on the right in (2) tends to zero since the numerator contains ρ and the denominator ρ^2. Thus the first term on the right tends to π (the value of the contour integral) as $\rho \to \infty$, and we have

$$I = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \pi$$

This method can be used to evaluate any integral of the form

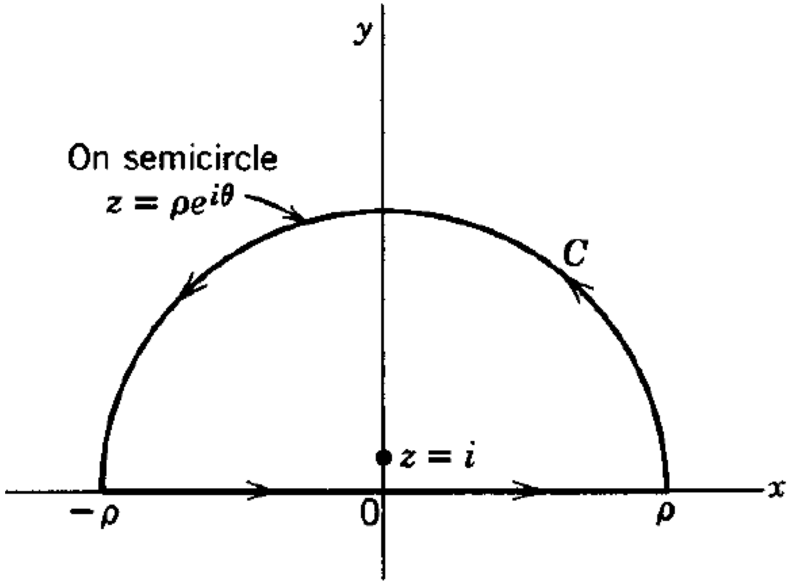
$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$$

if $P(x)$ and $Q(x)$ are polynomials with the degree of Q at least two greater than the degree of P, and if $Q(z)$ has no real zeros (that is, zeros on the x axis). If the integrand $P(x)/Q(x)$ is an even function, then we can also find the integral from 0 to ∞.

$$I = \int_0^\infty \frac{\cos x\,dx}{1+x^2}$$

We consider the contour integral

$$\oint_C \frac{e^{iz}\,dz}{1+z^2}$$



The singular point inclosed is again $z = i$, and the residue there is

$$\lim_{z\to i}(z-i)\frac{e^{iz}}{(z-i)(z+i)} = \frac{e^{-1}}{2i} = \frac{1}{2ie}$$

The value of the contour integral is $2\pi i(1/2ie) = \pi/e$

tends to zero as $\rho \to \infty$

$$\oint_C \frac{e^{iz}\,dz}{1+z^2} = \int_{-\rho}^{\rho} \frac{e^{ix}\,dx}{1+x^2} + \int_{\substack{\text{along upper half}\\ \text{of } z=\rho e^{i\theta}}} \frac{e^{iz}\,dz}{1+z^2}$$

$$|e^{iz}| = |e^{ix-y}| = |e^{ix}||e^{-y}| = e^{-y} \le 1$$

since $y \geq 0$ on the contour we are considering.

Since $|e^{iz}| \leq 1$,the integral along the semicircle tends to zero as the radius $\rho \to \infty$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = \frac{\pi}{e}$$

taking the real part of both sides of this equation

$$\int_{-\infty}^{\infty} \frac{\cos x\,dx}{1+x^2} = \frac{\pi}{e}$$

Since the integrand $(\cos x)/(1+x^2)$ is an even function,

$$\int_{0}^{\infty} \frac{\cos x\,dx}{1+x^2} = \frac{\pi}{2e}$$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{imx}\, dx = 2\pi i \cdot \text{sum of the residues of } \frac{P(z)}{Q(z)} e^{imz}$$

in the upper half-plane if all the following requirements are met:

1. $P(x)$ and $Q(x)$ are polynomials, and
2. $Q(x)$ has no real zeros, and
3. the degree of $Q(x)$ is at least 1 greater than the degree of $P(x)$, and $m > 0$.

$$\int \frac{P(z)}{Q(z)} e^{imz}\, dz \quad \text{around the semicircle tend to zero as } \rho \to \infty.$$

By taking real and imaginary parts, we then find the integrals

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos mx\, dx, \qquad \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin mx\, dx$$

11. $\displaystyle\int_0^\infty \frac{dx}{(4x^2+1)^3}$

12. $\displaystyle\int_0^\infty \frac{x^2\,dx}{x^4+16}$

13. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+4)(x^2+9)}$

14. $\displaystyle\int_{-\infty}^\infty \frac{\sin x\,dx}{x^2+4x+5}$

15. $\displaystyle\int_0^\infty \frac{\cos 2x\,dx}{9x^2+4}$

16. $\displaystyle\int_0^\infty \frac{x\sin x\,dx}{9x^2+4}$

17. $\displaystyle\int_{-\infty}^\infty \frac{x\sin x\,dx}{x^2+4x+5}$

18. $\displaystyle\int_0^\infty \frac{\cos \pi x\,dx}{1+x^2+x^4}$

19. $\displaystyle\int_0^\infty \frac{\cos 2x\,dx}{(4x^2+9)^2}$

20. $\displaystyle\int_0^\infty \frac{\cos x\,dx}{(1+9x^2)^2}$