5.11 Properties of the Differential Wave Equation

We have seen that the particle displacement ξ obeys the differential equation for plane waves, i.e.,

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2}. \tag{5.42}$$

ξ is a function of the independent variables x and t. We shall show here that the particle velocity, dilatation, condensation, acoustic pressure etc. satisfy the same differential wave equation as ξ and propagate with the same phase velocity c.

(i) The particle velocity u is given by

$$u = \frac{\partial \xi}{\partial t}. \tag{5.43}$$

Differentiating both sides of Eq. (5.42) with respect to t, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \xi}{\partial t^2}\right) = c^2 \frac{\partial}{\partial t}\left(\frac{\partial^2 \xi}{\partial x^2}\right). \tag{5.44}$$

Since x and t are independent variables, the order of differentiation with respect to x and t can be interchanged in Eq. (5.44) to yield

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial \xi}{\partial t}\right) = c^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial \xi}{\partial t}\right)$$

or,
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

which is the same as Eq. (5.42). Thus the particle velocity u obeys the wave equation. In a similar way, it is easily shown that the particle acceleration $\frac{\partial u}{\partial t} = \frac{\partial^2 \xi}{\partial t^2}$ satisfies the wave equation.

(ii) The dilatation in plane waves has been already shown to be given by

$$\Delta = \frac{\partial \xi}{\partial x} = -s \tag{5.45}$$

where s is the condensation. Differentiating Eq. (5.42) with respect to x gives

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 \xi}{\partial t^2}\right) = c^2 \frac{\partial}{\partial x}\left(\frac{\partial^2 \xi}{\partial x^2}\right)$$

$$\text{or,} \quad \frac{\partial^2}{\partial t^2}\left(\frac{\partial \xi}{\partial x}\right) = c^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial \xi}{\partial x}\right)$$

$$\text{or,} \quad \frac{\partial^2 \Delta}{\partial t^2} = c^2 \frac{\partial^2 \Delta}{\partial x^2}. \tag{5.46}$$

So, Δ (and hence s) obeys the wave equation.

(iii) The acoustic pressure p is related to the condensation s by $p = Ks$, where K is the bulk modulus. Since s satisfies the wave equation, it follows that p also satisfies the same, and so does the pressure gradient $\partial p/\partial x$. Furthermore, since the excess density is $\delta\rho = \rho_0 s$, obviously the excess density obeys the wave equation.

5.12 Simple Harmonic Solution of the Wave Equation

We have seen that the differential equation for the wave field parameter ψ associated with a plane wave propagating with the phase velocity c, is

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}. \tag{5.47}$$

The results of Sec. 5.11 show that apart from the particle displacement ξ, the particle velocity, dilatation, acoustic pressure etc. can serve as the wave field parameter. For small displacements when Hooke's law is valid, i.e., the restoring force is proportional to the displacement, the particle displacement ξ varies simple harmonically. Any of the forms given by Eqs. (5.16a) through (5.16d) for ξ is acceptable. By direct substitution, it is readily verified that these forms satisfy the differential wave equation.

Taking

$$\xi = A\cos(\omega t - kx). \tag{5.48}$$

similar harmonic solutions can be written for any of the other wave field parameters. For example, we write for the particle velocity

$$u = \frac{\partial \xi}{\partial t} = -\omega A \sin(\omega t - kx)$$

$$= \omega A \cos\left(\omega t - kx + \frac{\pi}{2}\right). \tag{5.49}$$

The dilatation is

$$\Delta = \frac{\partial \xi}{\partial x} = kA \sin(\omega t - kx) = kA \cos\left(\omega t - kx - \frac{\pi}{2}\right). \tag{5.50}$$

The condensation is

$$s = -\Delta = -kA \sin(\omega t - kx) = kA \cos\left(\omega t - kx + \frac{\pi}{2}\right). \tag{5.51}$$

The acoustic pressure is

$$p = Ks = -KkA \sin(\omega t - kx) = -c^2 \rho_0 kA \sin(\omega t - kx)$$

$$= c^2 \rho_0 kA \cos\left(\omega t - kx + \frac{\pi}{2}\right). \tag{5.52}$$

The excess density is

$$d = \rho_0 s = -\rho_0 kA \sin(\omega t - kx) = \rho_0 kA \cos\left(\omega t - kx + \frac{\pi}{2}\right). \tag{5.53}$$

The above relationships show that u, s, p and d are in phase and all of them lead the particle displacement ξ by $\pi/2$ in time. Only the dilatation Δ lags ξ by $\pi/2$ in time.

Observation

The maximum acoustic pressure is $c^2 \rho_0 kA$, where A is the maximum particle displacement. The plots of ξ and p against t for a particular value of x are shown in Fig. 5.5. The curves show the harmonic variations given by Eqs. (5.48) and (5.52). Because of the phase difference of $\pi/2$ between ξ and p, the excess pressure is zero at maximum or minimum displacement; the excess pressure is a maximum or a minimum when the displacement is zero.

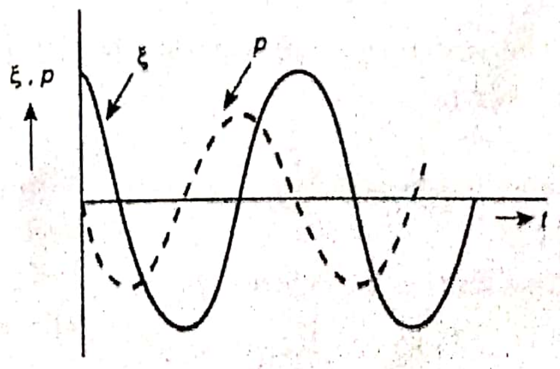

Fig. 5.5 Plots of ξ (solid line) and p (broken lines) against t for a monochromatic sound wave

The root-mean square sound pressure p_{rms} is given by

$$p_{rms}^2 = \frac{1}{T}\int_0^T p^2 dt = \frac{(c^2\rho_0 kA)^2}{T}\int_0^T \sin^2(\omega t - kx)dt$$

$$= \frac{(c^2\rho_0 kA)^2}{2},$$

so that $$p_{rms} = \frac{c^2\rho_0 kA}{\sqrt{2}} = \frac{\sqrt{2}\pi c^2\rho_0 A}{\lambda}. \quad (5.54)$$