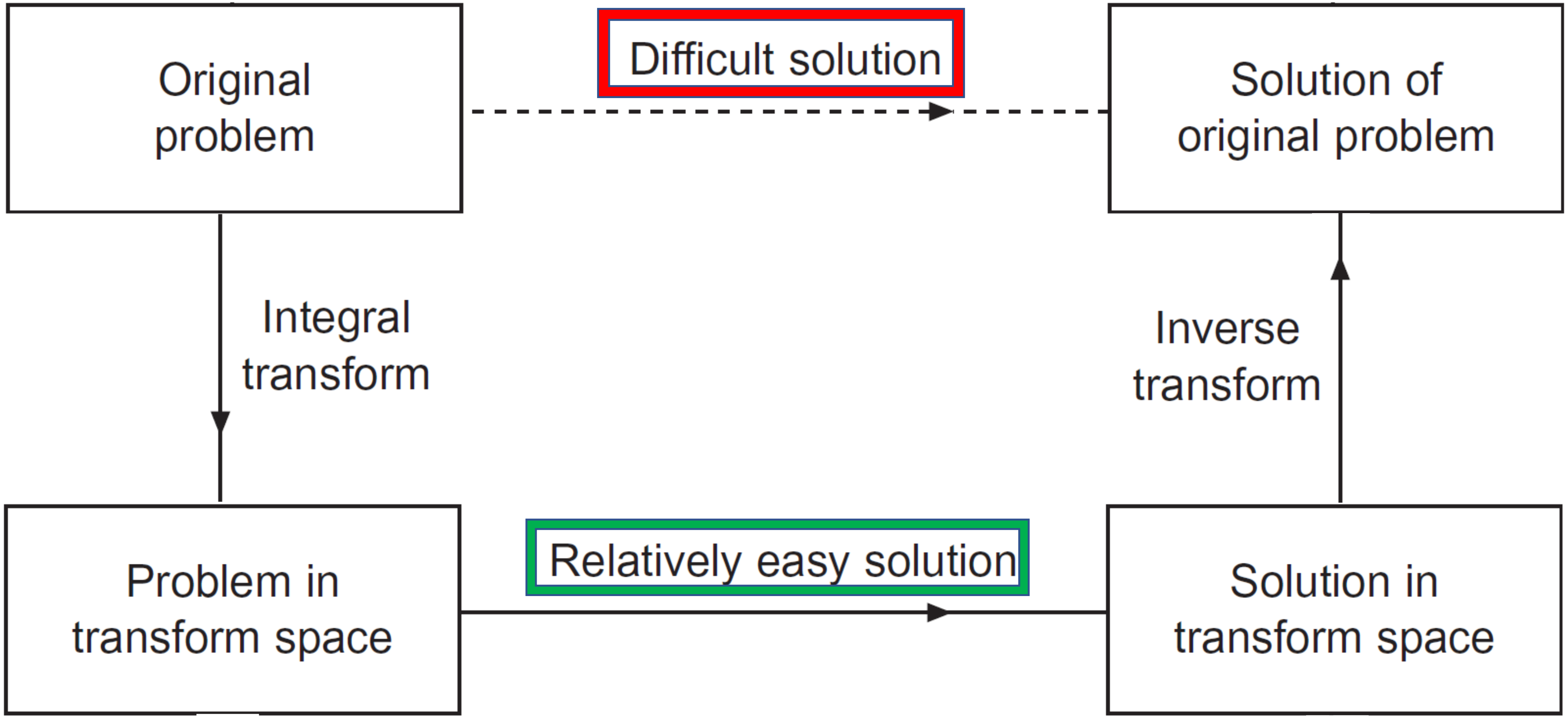
Fourier Transform

Acknowledgement

- Mathematical Methods for Physicists – Arfken, Weber & Harris


Original problem
Difficult solution
Solution of original problem
Integral transform
Inverse transform
Problem in transform space
Relatively easy solution
Solution in transform space

Fourier transform of $f(t)$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} d\omega$$

→ inverse Fourier transform

$$g_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \omega t \, dt$$ → Fourier cosine transform

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \omega t \, dt$$ → Fourier sine transform

Find the Fourier transform of

$$\boxed{f(t) = e^{-\alpha|t|}} \qquad \alpha > 0$$

$$g(\omega) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{0} e^{\alpha t + i\omega t}\, dt + \sqrt{\frac{1}{2\pi}} \int_{0}^{\infty} e^{-\alpha t + i\omega t}\, dt$$

$$= \sqrt{\frac{1}{2\pi}} \left[\frac{1}{\alpha + i\omega} + \frac{1}{\alpha - i\omega} \right] = \sqrt{\frac{1}{2\pi}} \frac{2\alpha}{\alpha^2 + \omega^2}$$

We note two features of this result:

(1) It is real; if $f(t)$ is even, its transform will be real.

(2) The more localized is $f(t)$, the less localized will be $g(\omega)$.

Find the Fourier transform of

$$\boxed{f(t) = \delta(t)}$$

$$g(\omega) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} \delta(t) e^{i\omega t}\, dt$$

$$= \sqrt{\frac{1}{2\pi}}$$

This is the ultimately localized $f(t)$, and we see that $g(\omega)$ is completely delocalized; it has the same value for all ω.

Fourier transform of Gaussian

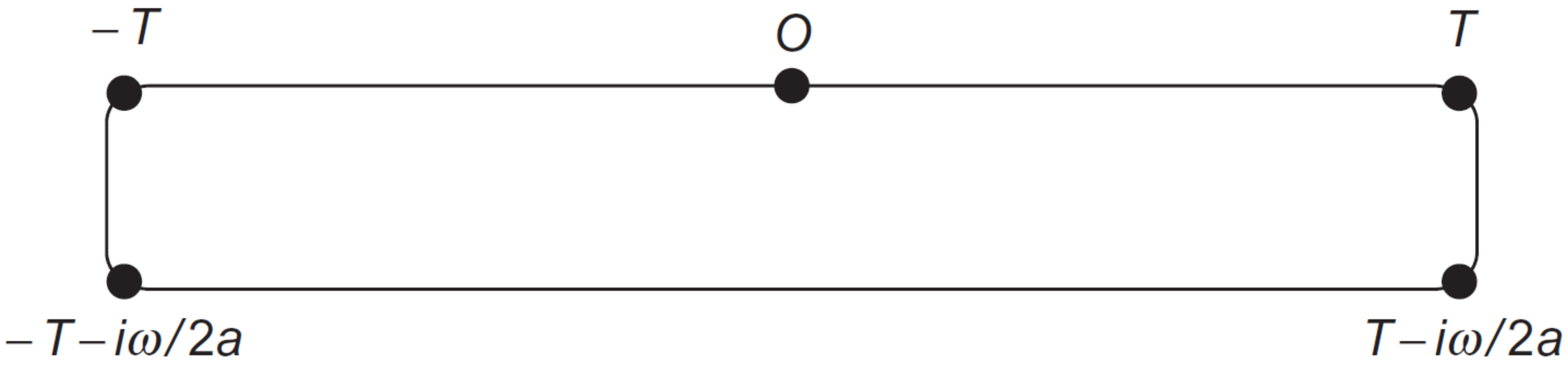
Gaussian function e^{-at^2} $\qquad a > 0$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-at^2} e^{i\omega t}\, dt$$

$$-at^2 + i\omega t = -a\left(t - \frac{i\omega}{2a}\right)^2 - \frac{\omega^2}{4a}$$

integration variable from t to $s = t - i\omega/2a$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/4a} \int_{-T-i\omega/2a}^{T-i\omega/2a} e^{-as^2} ds \qquad \text{(limit of large } T\text{)}$$



The s integration, is on a path parallel to, but below the real axis by an amount $i\omega/2a$. But because connections from that path to the real axis at $\pm T$ make negligible contributions to a contour integral and since the contours enclose no singularities, the integral is equivalent to one along the real axis. Changing the integration limits to $\pm\infty$ and rescaling to the new variable $\xi = s/\sqrt{a}$, we reach

$$\int_{-\infty}^{\infty} e^{-as^2} dt = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-\xi^2} d\xi$$

$$= \sqrt{\frac{\pi}{a}}$$

$$g(\omega) = \frac{1}{\sqrt{2a}} \exp\left(-\frac{\omega^2}{4a}\right) \Rightarrow \quad \text{Gaussian}(\omega\text{-space})$$

Fourier transform and their inverses

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, dt$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} \, d\omega$$

$$g_c(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \cos \omega t \, dt$$

$$f_c(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} g(\omega) \cos \omega t \, d\omega$$

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin \omega t \, dt$$

$$f_s(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} g(\omega) \sin \omega t \, d\omega$$

Fourier Transform (three-dimensional space)

$$g(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int f(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3r$$

amplitude of the wave $\exp(-i\mathbf{k}\cdot\mathbf{r})$

$$f(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int g(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3k$$

expansion of a function $f(\mathbf{r})$ in a continuum of plane waves.

Properties of Fourier Transform (Translation 3D & 1D)

$[\cdots]^T$ to denote the Fourier transform of the included object

$$\Big[f(\mathbf{r}-\mathbf{R})\Big]^T(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\int f(\mathbf{r}-\mathbf{R})e^{i\mathbf{k}\cdot\mathbf{r}}d^3r$$

$$= \frac{1}{(2\pi)^{3/2}}\int f(\mathbf{r})e^{i\mathbf{k}\cdot(\mathbf{r}+\mathbf{R})}d^3r$$

$$= e^{i\mathbf{k}\cdot\mathbf{R}}g(\mathbf{k})$$

$$\Big[f(t-a)\Big]^T(\omega) = e^{i\omega t}g(\omega)$$

Properties of Fourier Transform (Change of scale 3D & 1D)

$$\left[f(\alpha\mathbf{r})\right]^T = \frac{1}{(2\pi)^{3/2}} \int f(\alpha\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3r$$

$$= \frac{1}{(2\pi)^{3/2}} \int f(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}/\alpha} \alpha^{-3} d^3r$$

$$= \frac{1}{\alpha^3} g(\alpha^{-1}\mathbf{k})$$

$$\left[f(\alpha t)\right]^T(\omega) = \frac{1}{\alpha} g(\alpha^{-1}\omega)$$

Properties of Fourier Transform (Sign change 3D & 1D)

$$\Big[f(-\mathbf{r})\Big]^{T}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\int f(-\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}d^3r$$

$$= \frac{1}{(2\pi)^{3/2}}\int f(\mathbf{r})e^{i\mathbf{k}\cdot(-\mathbf{r})}d^3r$$

$$= g(-\mathbf{k})$$

$$\Big[f(-t)\Big]^{T}(\omega) = g(-\omega)$$

Properties of Fourier Transform (Complex Conjugation 3D & 1D)

$$\Big[f^*(-\mathbf{r})\Big]^T(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}}\int f^*(-\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}d^3r$$

$$= \left[\frac{1}{(2\pi)^{3/2}}\int f(-\mathbf{r})e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r\right]^*$$

$$= \left[\frac{1}{(2\pi)^{3/2}}\int f(\mathbf{r})e^{-i\mathbf{k}\cdot(-\mathbf{r})}d^3r\right]^*$$

$$= g^*(\mathbf{k})$$

$$\Big[f^*(-t)\Big]^T(\omega) = g^*(\omega)$$

Properties of Fourier Transform (Gradient 3D & 1D)

$$\nabla f(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int g(\mathbf{k}) \left[\nabla_r e^{-i\mathbf{k}\cdot\mathbf{r}} \right] d\mathbf{k}$$

$$= \frac{1}{(2\pi)^{3/2}} \int g(\mathbf{k}) \left[(-i\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} \right] d\mathbf{k}$$

$$= \frac{1}{(2\pi)^{3/2}} \int \left[-i\mathbf{k}\, g(\mathbf{k}) \right] e^{-i\mathbf{k}\cdot\mathbf{r}} \, d\mathbf{k}$$

$$\left[\nabla f(\mathbf{r}) \right]^T (\mathbf{k}) = -i\mathbf{k}\, g(\mathbf{k})$$

$$\left[f'(t) \right]^T (\omega) = -i\omega\, g(\omega) \qquad \text{(first derivative)}$$

Properties of Fourier Transform (Laplacian 3D & 1D)

$$\left[\nabla^2 f(\mathbf{r})\right]^T = \frac{1}{(2\pi)^{3/2}} \int \nabla^2 f(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3r$$

Applying Green's theorem

$$= \frac{1}{(2\pi)^{3/2}} \int f(\mathbf{r}) \nabla^2 e^{i\mathbf{k}\cdot\mathbf{r}}\, d^3r$$

$$\nabla^2 e^{i\mathbf{k}\cdot\mathbf{r}} \rightarrow \nabla^2 e^{ik_x x} e^{ik_y y} e^{ik_z z} \rightarrow (-k_x^2 - k_y^2 - k_z^2) e^{ik_x x} e^{ik_y y} e^{ik_z z} \Rightarrow -k^2 e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$\left[\nabla^2 f(\mathbf{r})\right]^T (\mathbf{k}) = -k^2 g(\mathbf{k})$$

1. The function $f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

is a symmetrical finite step function.

(a) Find $g_c(\omega)$, Fourier cosine transform of $f(x)$.

(b) Taking the inverse cosine transform, show that

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin\omega \cos\omega x}{\omega} \, d\omega.$$

(c) From part (b) show that $\displaystyle\int_0^\infty \frac{\sin\omega \cos\omega x}{\omega} d\omega = \begin{cases} 0, & |x| > 1, \\ \frac{\pi}{4}, & |x| = 1, \\ \frac{\pi}{2}, & |x| < 1. \end{cases}$

2.

(a) Show that the Fourier sine and cosine transforms of e^{-at} are

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \frac{\omega}{\omega^2 + a^2}, \qquad g_c(\omega) = \sqrt{\frac{2}{\pi}} \frac{a}{\omega^2 + a^2}.$$

(b) Show that

$$\int_0^\infty \frac{\omega \sin \omega x}{\omega^2 + a^2} d\omega = \frac{\pi}{2} e^{-ax}, \qquad x > 0,$$

$$\int_0^\infty \frac{\cos \omega x}{\omega^2 + a^2} d\omega = \frac{\pi}{2a} e^{-ax}, \qquad x > 0.$$

3. Find the Fourier transform of the triangular pulse

$$f(x) = \begin{cases} h(1 - a|x|), & |x| < 1/a, \\ 0, & |x| > 1/a. \end{cases}$$