Laplace Transforms

Acknowledgement

- Mathematical Methods for Physicists – Arfken, Weber & Harris

The Laplace transform $f(s)$ of a function $F(t)$ is defined by

$$f(s) = \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t) dt$$

$$F(t) = 1, \quad t > 0$$

The Laplace transform becomes

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt$$

$$= \frac{1}{s} \qquad \text{for} \quad s > 0$$

$$F(t) = e^{kt}, \quad t > 0$$

The Laplace transform becomes

$$\mathcal{L}\left\{e^{kt}\right\} = \int_0^\infty e^{-st} e^{kt}\, dt$$

$$= \frac{1}{s-k} \qquad \text{for} \quad s > k$$

$$\cosh kt = \tfrac{1}{2}(e^{kt} + e^{-kt})$$

$$\mathcal{L}\{\cosh kt\} = \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right)$$

$$= \frac{s}{s^2 - k^2} \qquad \text{for} \quad s > k$$

$$\sinh kt = \tfrac{1}{2}(e^{kt} - e^{-kt})$$

$$\mathcal{L}\{\sinh kt\} = \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right)$$

$$= \frac{k}{s^2 - k^2} \qquad \text{for} \quad s > k$$

$$\cos kt = \cosh ikt$$

Laplace Transform

$$\frac{s}{s^2 - k^2}$$

k is replaced by ik

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2 + k^2}$$

$$\sin kt = -i \sinh ikt$$

$$\frac{k}{s^2 - k^2}$$

k is replaced by ik

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2 + k^2}$$

$$F(t) = t^n$$

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n \, dt$$

gamma function

$$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} \qquad s > 0, \; n > -1$$

Heaviside Step Function

$$u(t-k)=\begin{cases}0, & t<k\\ 1, & t>k\end{cases}$$

The Laplace transform becomes

$$\mathcal{L}\{u(t-k)\}=\int_k^{\infty} e^{-st}\,dt$$

$$=\frac{1}{s}e^{-ks}$$

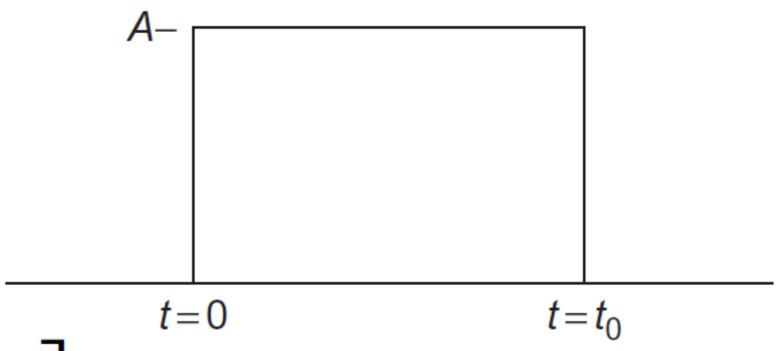
Laplace Transform of Square Pulse

Let's compute the transform of a square pulse $F(t)$ of height A

from $t = 0$ to $t = t_0$

Using the Heaviside step function,

the pulse can be represented as



$$F(t) = A\Big[u(t) - u(t - t_0)\Big]$$

$$\mathcal{L}\{F(t)\} = \frac{1}{s}\left(1 - e^{-t_0 s}\right)$$

Dirac Delta Function

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0)dt$$

$$= e^{-st_0} \qquad \text{for} \quad t_0 > 0$$

$$\mathcal{L}\{\delta(t)\} = 1$$

This delta function is frequently called the **impulse** function because it is so useful in describing impulsive forces.

Inverse Laplace Transform

$$\mathcal{L}\{F(t)\} = f(s)$$

The inverse transform

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

A table of transforms can be built up and used to identify inverse transformations, exactly as a table of logarithms can be used to look up antilogarithms.

Table of Laplace Transforms

	$f(s)$	$F(t)$	Limitation
1.	1	$\delta(t)$	Singularity at $+0$
2.	$\frac{1}{s}$	1	$s > 0$
3.	$\frac{\Gamma(n+1)}{s^{n+1}}$	t^n	$s > 0,\ n > -1$
4.	$\frac{1}{s-k}$	e^{kt}	$s > k$
5.	$\frac{1}{(s-k)^2}$	te^{kt}	$s > k$
6.	$\frac{s}{s^2-k^2}$	$\cosh kt$	$s > k$

	$f(s)$	$F(t)$	Limitation
7.	$\dfrac{k}{s^2 - k^2}$	$\sinh kt$	$s > k$
8.	$\dfrac{s}{s^2 + k^2}$	$\cos kt$	$s > 0$
9.	$\dfrac{k}{s^2 + k^2}$	$\sin kt$	$s > 0$
10.	$\dfrac{s-a}{(s-a)^2 + k^2}$	$e^{at} \cos kt$	$s > a$
11.	$\dfrac{k}{(s-a)^2 + k^2}$	$e^{at} \sin kt$	$s > a$
12.	$\dfrac{s^2 - k^2}{(s^2 + k^2)^2}$	$t \cos kt$	$s > 0$
13.	$\dfrac{2ks}{(s^2 + k^2)^2}$	$t \sin kt$	$s > 0$

Partial Fraction Expansion

The function $f(s) = k^2/s(s^2+k^2)$ does not appear as a transform listed in table, but we may obtain it from the tabulated transforms by observing that it has the partial fraction expansion

$$f(s) = \frac{k^2}{s(s^2+k^2)}$$

$$= \frac{1}{s} - \frac{s}{s^2+k^2}$$

Each of the two partial fractions corresponds to an entry in table and we can therefore take the inverse transform of $f(s)$ term by term:

$$\mathcal{L}^{-1}\{f(s)\} = 1 - \cos kt$$

1. Verify that

$$\mathcal{L}\left\{\frac{\cos at - \cos bt}{b^2 - a^2}\right\} = \frac{s}{(s^2 + a^2)(s^2 + b^2)}, \qquad a^2 \neq b^2$$

2. Using partial fraction expansions, show that

(a) $$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = \frac{e^{-at} - e^{-bt}}{b-a}, \quad a \neq b.$$

(b) $$\mathcal{L}^{-1}\left\{\frac{s}{(s+a)(s+b)}\right\} = \frac{ae^{-at} - be^{-bt}}{a-b}, \quad a \neq b$$

3. Using partial fraction expansions, show that for $a^2 \neq b^2$,

(a) $$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)(s^2+b^2)}\right\} = -\frac{1}{a^2-b^2}\left\{\frac{\sin at}{a} - \frac{\sin bt}{b}\right\}$$

(b) $$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \frac{1}{a^2-b^2}\{a\sin at - b\sin bt\}$$