Laplace Transform

Acknowledgement

- Mathematical Methods for Physicists – Arfken, Weber & Harris

Laplace Transform of Derivatives

Let us transform the first derivative of $F(t)$:

$$\mathcal{L}\{F'(t)\} = \int_0^\infty e^{-st} \frac{dF(t)}{dt} dt$$

Integrating by parts, we obtain

$$\mathcal{L}\{F'(t)\} = e^{-st} F(t) \Big|_0^\infty + s \int_0^\infty e^{-st} F(t) dt$$

$$= s\mathcal{L}\{F(t)\} - F(0)$$

$$\mathcal{L}\left\{F'(t)\right\} = s\mathcal{L}\{F(t)\} - F(0)$$

Strictly speaking, $F(0) = F(+0)$

An extension to higher derivatives gives

$$\mathcal{L}\left\{F^{(2)}(t)\right\} = s^2\mathcal{L}\{F(t)\} - sF(+0) - F'(+0)$$

$$\mathcal{L}\{F^{(n)}(t)\} = s^n\mathcal{L}\{F(t)\} - s^{n-1}F(+0) - \cdots - F^{(n-1)}(+0)$$

The Laplace transform, like the Fourier transform, replaces differentiation with multiplication.

Change of Scale

If we replace t by at in the defining formula for the Laplace transform,

$$\mathcal{L}\{F(at)\} = \int_0^\infty e^{-st} F(at)dt$$

$$= \frac{1}{a}\int_0^\infty e^{-(s/a)(at)} F(at)\, d(at)$$

$$= \frac{1}{a} f\left(\frac{s}{a}\right)$$

Substitution

If we replace the parameter s by $s - a$ in the definition of the Laplace transform, we have

$$f(s-a) = \int_0^\infty e^{-(s-a)t} F(t)dt$$

$$= \int_0^\infty e^{-st} e^{at} F(t)dt$$

$$= \mathcal{L}\left\{e^{at} F(t)\right\}$$

$$f(s-a) = \mathcal{L}\left\{e^{at} F(t)\right\}$$

Hence the replacement of s with $s - a$ corresponds to multiplying $F(t)$ by e^{at}, and conversely.

$$\mathcal{L}\left\{e^{at} \sin kt\right\} = \frac{k}{(s-a)^2 + k^2}, \quad (s > a)$$

$$\mathcal{L}\left\{e^{at} \cos kt\right\} = \frac{s-a}{(s-a)^2 + k^2}, \quad s > a$$

Simple Harmonic Oscillator

$$m\frac{d^2X(t)}{dt^2} + kX(t) = 0$$

initial conditions

$$X(0) = X_0,$$
$$X'(0) = 0.$$

Applying the Laplace transform, we obtain

$$m\mathcal{L}\left\{\frac{d^2X}{dt^2}\right\} + k\mathcal{L}\{X(t)\} = 0$$

Letting $x(s)$ denote the presently unknown transform $\mathcal{L}\{X(t)\}$

$$ms^2x(s) - msX_0 + kx(s) = 0$$

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Solving algebraic Equation

$$x(s) = X_0 \frac{s}{s^2 + \omega_0^2}$$

$$\omega_0^2 \equiv \frac{k}{m}$$

INVERSE LAPLACE TRANSFORM

$$X(t) = X_0 \cos \omega_0 t$$

Damped Oscillator

$$mX''(t) + bX'(t) + kX(t) = 0$$

initial conditions
$X(0) = X_0,$
$X'(0) = 0.$

Applying the Laplace transform, we obtain

$$m[s^2x(s) - sX_0] + b[sx(s) - X_0] + kx(s) = 0$$

Solving algebraic Equation

$$x(s) = X_0 \frac{ms + b}{ms^2 + bs + k}$$

$$x(s) = X_0 \frac{ms + b}{ms^2 + bs + k}$$

$$s^2 + \frac{b}{m}s + \frac{k}{m} = \left(s + \frac{b}{2m}\right)^2 + \left(\frac{k}{m} - \frac{b^2}{4m^2}\right)$$

Considering that the damping is small enough that $b^2 < 4km$, then the last term is positive and will be denoted by ω_1^2.

$$x(s) = X_0 \frac{s + b/m}{(s + b/2m)^2 + \omega_1^2}$$

$$x(s) = X_0 \frac{s + b/m}{(s + b/2m)^2 + \omega_1^2}$$

$$x(s) = X_0 \frac{s + b/2m}{(s + b/2m)^2 + \omega_1^2} + X_0 \frac{\omega_1 (b/2m\omega_1)}{(s + b/2m)^2 + \omega_1^2}$$

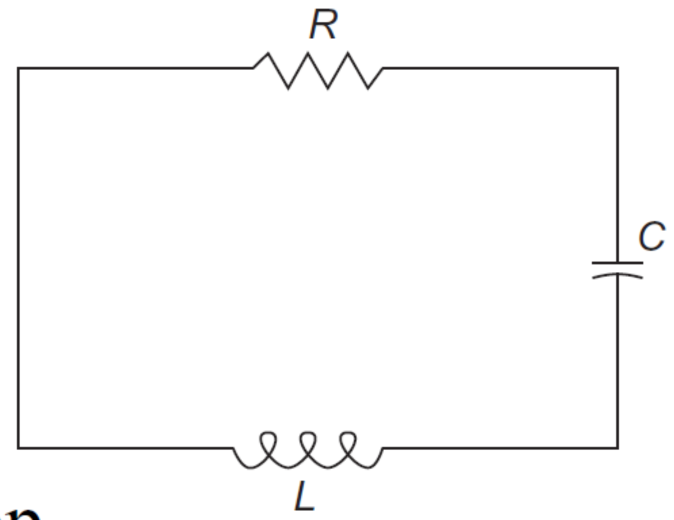
INVERSE LAPLACE TRANSFORM

$$X(t) = X_0 e^{-(b/2m)t} \left(\cos \omega_1 t + \frac{b}{2m\omega_1} \sin \omega_1 t \right)$$

$$X(t) = X_0 \frac{\omega_0}{\omega_1} e^{-(b/2m)t} \cos(\omega_1 t - \varphi) \quad \tan \varphi = \frac{b}{2m\omega_1}, \quad \omega_0^2 = \frac{k}{m}$$

RLC Circuit

It is worth noting the similarity between the damped simple harmonic oscillation of a mass and an *RLC* circuit.



The sum of the potential differences around the loop must be zero (Kirchhoff's law, conservation of energy). This gives

charge of capacitor

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int^{t} I\,dt = 0$$

$$L\frac{dI}{dt} + RI + \frac{1}{C}\int^{t} I\,dt = 0$$

Differentiating with respect to time (to eliminate the integral), we have

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0$$

If we replace $I(t)$ with $X(t)$, L with m, R with b, and C^{-1} with k, then it is identical with the mechanical problem.

Heaviside Shifting Theorem

This time let $f(s)$ be multiplied by e^{-bs}, with $b > 0$:

$$e^{-bs} f(s) = e^{-bs} \int_0^\infty e^{-st} F(t)dt$$

$$= \int_0^\infty e^{-s(t+b)} F(t)dt$$

Now let $t + b = \tau$

$$e^{-bs} f(s) = \int_b^\infty e^{-s\tau} F(\tau - b)d\tau$$

$$e^{-bs} f(s) = \int_{b}^{\infty} e^{-s\tau} F(\tau - b) d\tau$$

Since $F(t)$ is assumed to be equal to zero for $t < 0$, so that $F(\tau - b) = 0$ for $0 \le \tau < b$, we can change the lower limit to zero without changing the value of the integral. Then renaming τ as our standard Laplace transform variable t, we have

$$e^{-bs} f(s) = \int_{0}^{\infty} e^{-s\tau} F(\tau - b) d\tau$$

$$\boxed{e^{-bs} f(s) = \mathcal{L}\{F(t - b)\}}$$

Step Function

$$F(t) = \int_0^\infty \frac{\sin tx}{x} dx$$

Laplace transform of this definite (and improper) integral

$$f(s) = \mathcal{L}\left\{ \int_0^\infty \frac{\sin tx}{x} dx \right\}$$

$$= \int_0^\infty e^{-st} \int_0^\infty \frac{\sin tx}{x} dx\, dt$$

$$f(s) = \int_0^\infty e^{-st} \int_0^\infty \frac{\sin tx}{x} \, dx \, dt$$

Now, interchanging the order of integration, we get

Laplace transform of $\sin tx$

$$f(s) = \int_0^\infty \frac{1}{x} \left[\int_0^\infty e^{-st} \sin tx \, dt \right] dx$$

$$= \int_0^\infty \frac{dx}{s^2 + x^2}$$

$$f(s) = \int_0^\infty \frac{dx}{s^2 + x^2}$$

$$= \frac{1}{s} \tan^{-1}\left(\frac{x}{s}\right)\Big|_0^\infty$$

$$f(s) = \frac{\pi}{2s}$$

We carry out the inverse transformation to obtain

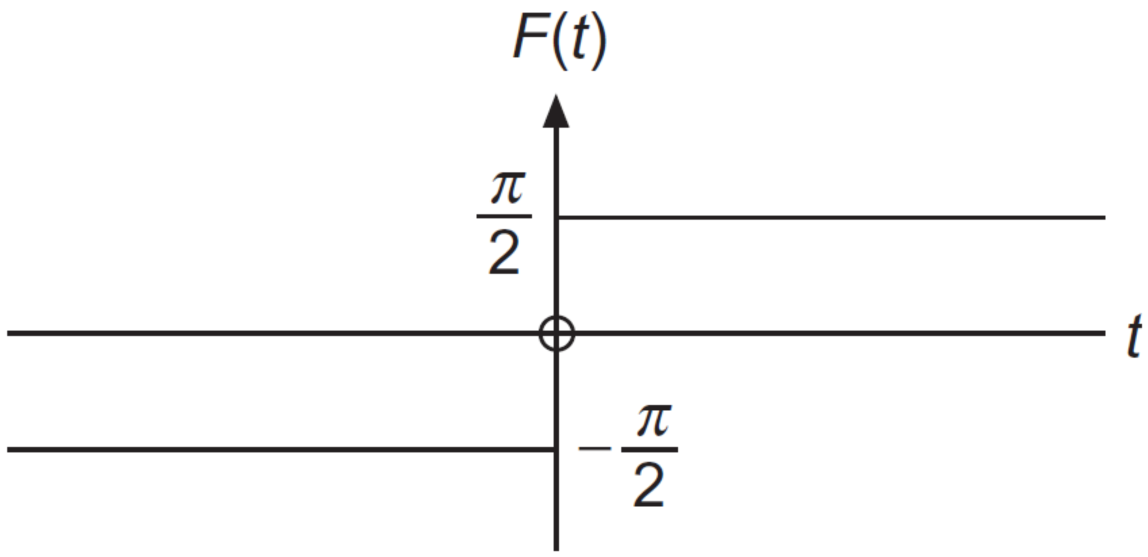
$$F(t) = \frac{\pi}{2}, \quad t > 0$$

For $F(-t)$ we need note only that $\sin(-tx) = -\sin tx$, giving $F(-t) = -F(t)$. Finally, if $t = 0$, $F(0)$ is clearly zero. Therefore,

$$\int_0^\infty \frac{\sin tx}{x} dx = \frac{\pi}{2}[2u(t) - 1] = \begin{cases} \frac{\pi}{2}, & t > 0 \\ 0, & t = 0 \\ -\frac{\pi}{2}, & t < 0. \end{cases}$$

Here $u(t)$ is the Heaviside unit step function·

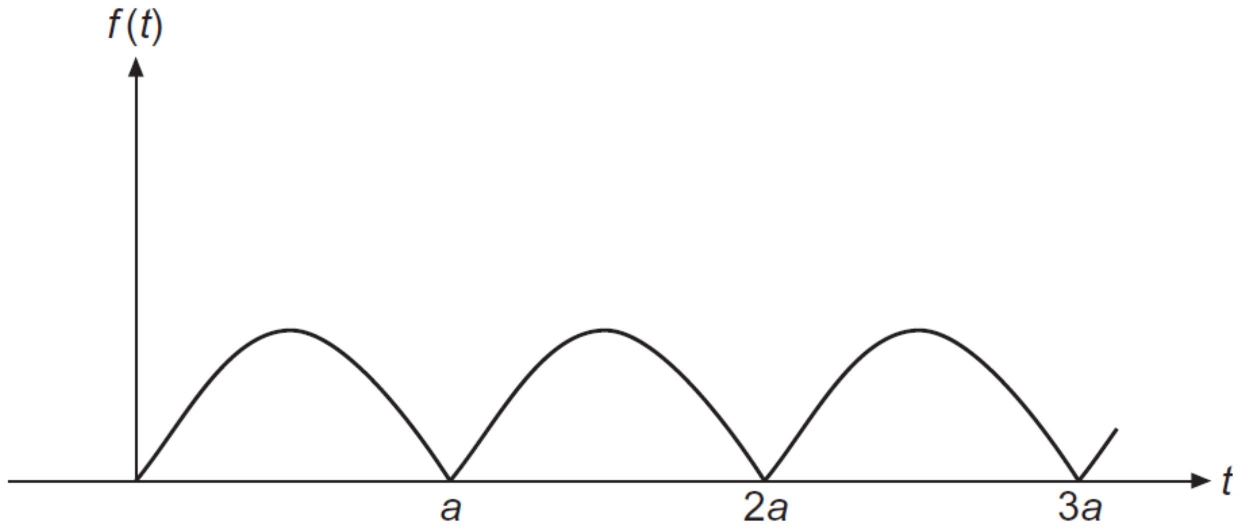
Thus, $\int_0^\infty (\sin tx/x)dx$, taken as a function of t, describes a step function with a step of height π at $t = 0$.



$$F(t) = \int_0^\infty \frac{\sin tx}{x} dx, \text{ a step function}$$

Periodic Function

$F(t)$ is periodic with a period a so that $F(t+a) = F(t)$ for all $t \geq 0$



$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

$$\mathcal{L}\{F(t)\} = \sum_{n=0}^{\infty} \int_{na}^{(n+1)a} e^{-st} F(t)\, dt$$

$$= \sum_{n=0}^{\infty} e^{-nas} \int_{0}^{a} e^{-st} F(t)\, dt$$

Performing the summation,

$$\mathcal{L}\{F(t)\} = \frac{1}{1 - e^{-as}} \int_{0}^{a} e^{-st} F(t)\, dt$$

Note that the integration is now over only the **first period** of $F(t)$.

Derivative of a Transform

$$f(s) = \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)dt$$

differentiated with respect to s

$$f'(s) = \int_0^\infty (-t)e^{-st} F(t)dt = \mathcal{L}\{-tF(t)\}$$

Continuing this process, we obtain

$$f^{(n)}(s) = \mathcal{L}\left\{(-t)^n F(t)\right\}$$

$$\mathcal{L}\left\{e^{kt}\right\} = \int_0^\infty e^{-st} e^{kt}\, dt = \frac{1}{s-k}, \quad s > k$$

Differentiating with respect to s (or with respect to k), we obtain

$$\mathcal{L}\left\{te^{kt}\right\} = \frac{1}{(s-k)^2}, \quad s > k$$

If we replace k by ik and separate into its real and imaginary parts,

$$\mathcal{L}\{t \cos kt\} = \frac{s^2 - k^2}{(s^2 + k^2)^2} \qquad \mathcal{L}\{t \sin kt\} = \frac{2ks}{(s^2 + k^2)^2}$$

Integration of Transforms

$$f(x) = \int_0^\infty e^{-xt} F(t)dt$$

Now reversing the order of integration in the following equation:

$$\int_s^\infty f(x)dx = \int_s^\infty dx \int_0^\infty dt\, e^{-xt} F(t)$$

$$= \int_0^\infty e^{-st} \frac{F(t)}{t} dt \quad = \mathcal{L}\left\{\frac{F(t)}{t}\right\}$$

Laplace Convolution Theorem

We take two transforms,

$$f_1(s) = \mathcal{L}\{F_1(t)\} \quad \text{and} \quad f_2(s) = \mathcal{L}\{F_2(t)\},$$

and multiply them together:

$$f_1(s)\, f_2(s) = \int_0^\infty e^{-sx} F_1(x) dx \int_0^\infty e^{-sy} F_2(y) dy$$

If we introduce the new variable $t = x + y$ and integrate over t and y instead of x and y, the limits of integration become $(0 \le t \le \infty)$, $(0 \le y \le t)$.

$$f_1(s) f_2(s) = \int_0^\infty e^{-st}\, dt \int_0^t F_1(t-y) F_2(y) dy$$

$$= \mathcal{L}\left\{ \int_0^t F_1(t-y) F_2(y) dy \right\}$$

$$= \mathcal{L}\{F_1 * F_2\}$$

$$f_1(s)\, f_2(s) = \mathcal{L}\{F_1 * F_2\}$$

convolution of F_1 and F_2

$$\int_0^t F_1(t-z)\,F_2(z)\,dz \equiv F_1 * F_2$$

$$F_1 * F_2 = F_2 * F_1$$ ➡ convolution is symmetric

Carrying out the inverse transform, we also find

$$\mathcal{L}^{-1}\{f_1(s)\, f_2(s)\} = \int_0^t F_1(t-z)\,F_2(z)\,dz$$

$$= F_1 * F_2$$

1.

The motion of a body falling in a resisting medium may be described by

$$m\frac{d^2X(t)}{dt^2} = mg - b\frac{dX(t)}{dt}$$

when the retarding force is proportional to the velocity.

Find $X(t)$ and $dX(t)/dt$ for the initial conditions

$$X(0) = \left.\frac{dX}{dt}\right|_{t=0} = 0.$$

2.

Find the Laplace transform of the square wave (period a) defined by

$$F(t) = \begin{cases} 1, & 0 < t < a/2, \\ 0, & a/2 < t < a. \end{cases}$$

3.

Show that

(a) $$\mathcal{L}\{\cosh at \cos at\} = \frac{s^3}{s^4 + 4a^4}$$

(b) $$\mathcal{L}\{\cosh at \sin at\} = \frac{as^2 + 2a^3}{s^4 + 4a^4}$$

4.

From the convolution theorem show that

$$\frac{1}{s} f(s) = \mathcal{L}\left\{ \int_0^t F(x)dx \right\},$$

where $f(s) = \mathcal{L}\{F(t)\}$.